A Derivation of Symmetrical Component Theory and Symmetrical Component Networks

John Horak, Basler Electric Company

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This paper provides a review of some of the theory behind symmetrical component analysis and derives some of the basic calculations utilized in power system analysis for short circuit and open phase conditions. The paper starts with a review of the concepts of system impedances in the phase (ABC) domain and develops the three phase, two port, voltage drop equation, $\mathbf{V_S} - \mathbf{V_R} = \mathbf{Z} \cdot \mathbf{I}$, in substantial detail. Then, the paper reviews symmetrical component (012) domain theory. The paper shows the conversion of the ABC domain voltage drop equation to the equivalent 012 domain voltage drop equation and, in the process, correlates the terms Z_S, Z_M, Z_0, Z_1, Z_2, and other impedances and presents the relationship between the $\mathbf{Z_{ABC}}$ and $\mathbf{Z_{012}}$ impedance matrices. Next, the paper shows how the $\mathbf{Z_{ABC}}$ and $\mathbf{Z_{012}}$ matrices have a simplified form for certain symmetrical impedance networks. Then, the paper applies the ABC and 012 forms of the voltage drop equation to some of the more common basic fault types in order to show the development of the associated symmetrical component networks used for fault analysis. The analysis process that is developed allows for the development of the equations for faults not easily shown in classical symmetrical component networks, and some of these equations are shown in an appendix.

Basic Impedance Concepts

Many papers and texts on power system analysis use lumped impedance concepts and models and start from models such as figure 3. Let us back up, however, with a development of figure 3 in order to have a better foundation for our analysis. Going back to basic electromagnetic theory, recall that around a conductive loop, with changing flux ϕ inside the loop due to current i(t):

$$\begin{aligned} v(t) &= Ri(t) + \frac{d}{dt}\phi(t) \\ &= Ri(t) + L\frac{d}{dt}i(t) \end{aligned} \tag{1}$$

If only fundamental frequency voltages and currents are involved, this equation can be stated in terms of the phasor math equation:

$$V_{AC,FUND} = RI_{AC,FUND} + jX_L I_{AC,FUND} \tag{2}$$

where $X_L = \omega L$. In the rest of this paper the subscript "AC,FUND" will be implied and not explicitly stated, and the "L" subscript also will be generally implied without saying, so the equation reduces to:

$$V = RI + jXI \tag{3}$$

Examine figure 1, where the system becomes a bit more involved, with two voltages and two currents. The system shown is part of a larger system, so it is assumed to be possible that

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currents in the two sides of the loop are not the same and, hence, the figure and analysis includes allowance for $I_A \neq I_N$. The equations in the loop become:

$$v_{SENT}(t) - v_{RECEIVED}(t) = R_A i_A(t) + R_N i_N(t) + \frac{d}{dt}\phi(t) \qquad (4)$$
$$\phi = L_A i_A(t) + L_N i_N(t)$$

Sent and Received will be indicated by S and R in the balance of the paper. In phasor analysis this equation is stated as:

$$V_S - V_R = R_A I_A + R_N I_N + j\left(X_A I_A + X_N I_N \right) \qquad (5)$$

The level of flux that is induced in the loop is dependent on the dimensions of the loop (i.e., the cross sectional area of the loop) and the permeability of the material inside the loop (e.g., steel vs. air).

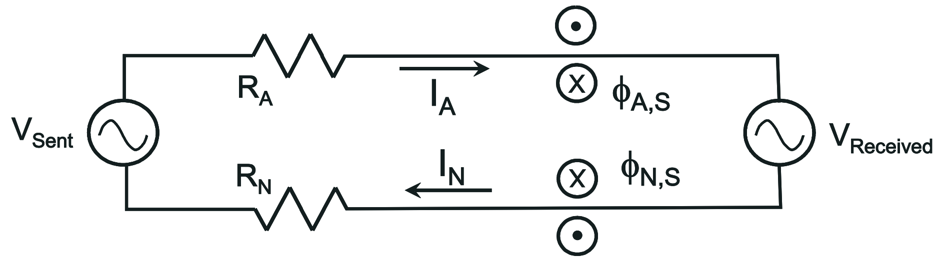

Figure 1 - Basic Analysis - Voltage in a Single Loop

In a three phase system, figure 2, the definition of L and X_L becomes substantially more complex since there are more conductors and flux loops and, hence, the flux distribution and the current return path is harder to define. In figure 2 there are three phase current loops, A, B, and C, each passing through a common neutral. The phase A loop is shown via a dotted line. Each phase loop in figure 2 has a different impedance. Each is defined by a different current path, a different loop cross section area and, when magnetic core material is involved, a different permeability of the material through which the flux passes. To keep the drawing from becoming exceedingly complex, only the main representative flux loops are shown and only for phase A. One's imagination should be used to fill in the blanks.

Current in each loop is assumed to return through the neutral conductor. Since I_N is a summation of I_A, I_B, and I_C, the voltage drop equations eventually will be reduced to remove I_N. Note the assumed direction for positive neutral current. In some text and articles, the reference direction for I_N is reversed, and the analysis is configured in such a manner that negative signs appear in some of the impedances to follow. In the development below, no negative signs appear in the impedances.

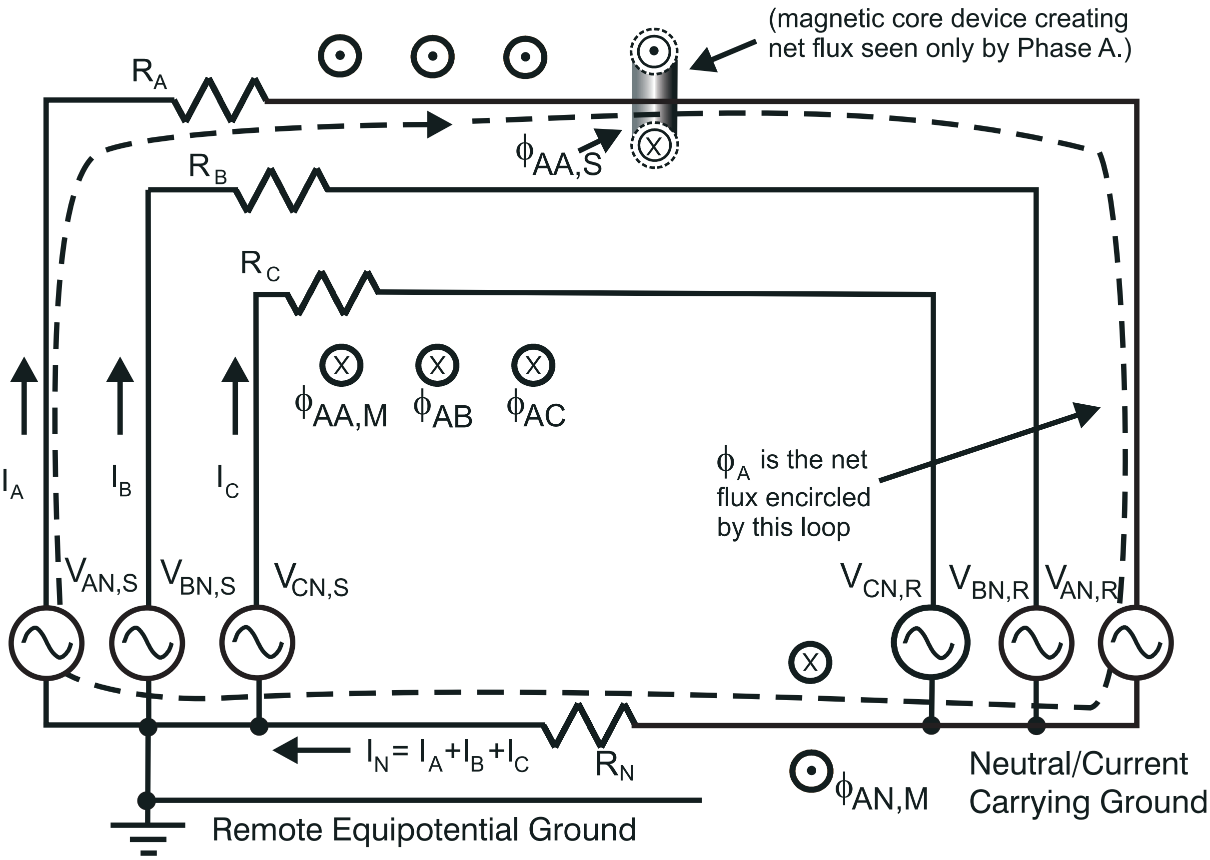

Figure 2 - Phase A Voltage Loop and Flux when Part of a Three Phase System

To understand the system impedances that will represent the system in figure 2, one should first understand the sources of flux that can be found in loop A in figure 2. The flux in loop A is the summation of the flux generated by the 4 currents specified in figure 2:

$$\phi_A = \phi_{AA} + \phi_{AB} + \phi_{AC} + \phi_{AN} \tag{6}$$

ϕ_{AA} = $\phi_{AA,S} + \phi_{AA,M}$

$\phi_{AA,S}$ = Flux due to I_A that is only seen by loop A. Consider this to be a “self” flux. Large self flux can be generated by magnetic core devices. The magnetic core device shown in the diagram has flux that is seen only by loop A and not by loop B or C.

$\phi_{AA,M}$ = Flux in loop A that is due to I_A, but that is also seen by loop B and/or C. This flux is the source of mutual inductance of phase A to phase B and C. The mutual flux to each loop B and C would, in practice, be slightly different.

ϕ_{AB} = Flux in loop A due to current I_B. This is the source of mutual coupling between I_B and phase A.

ϕ_{AC} = Flux in loop A due to current I_C. This is the source of mutual coupling between I_C and phase A.

ϕ_{AN} = Flux in loop A due to current I_N. This is the source of mutual coupling between I_N and phase A.

This pattern will exist for the loop associated with each phase.

The I_N flux distribution is a bit different than the I_A, I_B, and I_C flux distribution in that, in the elemental circuit of figure 2, there is no pure self-flux ($\phi_{NN,S}$) generated by current I_N. In an even more complete drawing, not shown in figure 2, the current in I_N will flow in a slightly differently located effective neutral conductor in the earth depending on whether the source for I_N is I_A, I_B, or I_C, which will result in varying Φ_{AN} for each source of I_N and introducing a small amount of $\phi_{NN,S}$. However, if we can accept the assumption of a single common neutral return path as "close enough," if one studies figure 2, the only way to have flux generated by I_N that does not pass through at least one of the A, B, or C phase loops is for the flux to be completely outside of the A, B, and C loops. If flux is outside the A, B, and C loops, then the flux that is generated will not be seen by the voltage sources.

The flux distribution (6) indicated by figure 2 can be restated in terms of L neutral current and can be removed from the equations:

$$\begin{aligned}\phi_A &= L_{AA} i_A(t) + L_{AB} i_B(t) + L_{AC} i_C(t) + L_{AN} i_N(t) \\ &= L_{AA} i_A(t) + L_{AB} i_B(t) + L_{AC} i_C(t) + L_{AN}\left(i_A(t) + i_B(t) + i_C(t)\right) \\ &= \left(L_{AA} + L_{AN}\right) i_A(t) + \left(L_{AB} + L_{AN}\right) i_B(t) + \left(L_{AC} + L_{AN}\right) i_C(t)\end{aligned} \tag{7}$$

This flux distribution, when coupled with resistive voltage drop, becomes the basis for system impedances.

There are two basic forms of impedance: "self" impedance and "mutual" impedance. Self impedance refers to the voltage drop in a phase loop due to current in that same phase. For instance, Z_{AA} relates the voltage loss in loop A due to I_A. The instantaneous voltage equation for current only in phase A is:

$$v_{AN,S}(t) - v_{AN,R}(t) = \left(R_A + R_N\right) i_A(t) + \left(L_{AA} + L_{AN}\right)\frac{d}{dt} i_A(t) \tag{8}$$

Note that the "N" reference is added to the voltage terms to clarify that the voltage is in reference to the local generator neutral and that voltage is not in reference to a perfect remote ground. Restating (8) in phasor analysis terms:

$$V_{AN,S} - V_{AN,R} = \left(R_A + R_N\right) I_A + j\left(X_{AA} + X_{AN}\right) I_A \tag{9}$$

The equation for self impedance becomes:

$$\begin{aligned}Z_{AA} &= \frac{V_{AN,S} - V_{AN,R}}{I_A} \\ &= \left(R_A + R_N\right) + j\left(X_{AA} + X_{AN}\right)\end{aligned} \tag{10}$$

Mutual impedance refers to the voltage drop in a loop due to current in another phase. For instance, Z_{AB} relates the voltage loss in loop A due to I_B. The instantaneous voltage equation for voltage induced in the phase A loop for current I_B is:

$$v_{AN,S}(t) - v_{AN,R}(t) = R_N i_B(t) + \left(L_{AB} + L_{AN}\right)\frac{d}{dt} i_B(t) \tag{11}$$

which in phasor analysis terms becomes restated to:

$$V_{AN,S} - V_{AN,R} = R_N I_B + j\left(X_{AB} + X_{AN} \right) I_B \tag{12}$$

So the equation for mutual impedance becomes:

$$\begin{aligned} Z_{AB} &= \frac{V_{AN,S} - V_{AN,R}}{I_B} \\ &= \left(R_N \right) + j\left(X_{AB} + X_{AN} \right) \end{aligned} \tag{13}$$

Now we can define the voltage drop equations that describe the circuit in figure 2:

$$\begin{aligned} &\begin{bmatrix} V_{AN,S} \\ V_{BN,S} \\ V_{CN,S} \end{bmatrix} - \begin{bmatrix} V_{AN,R} \\ V_{BN,R} \\ V_{CN,R} \end{bmatrix} = \\ &\begin{bmatrix} R_A + R_N + j\left(X_{AA} + X_{AN} \right) & R_N + j\left(X_{AB} + X_{AN} \right) & R_N + j\left(X_{AC} + X_{AN} \right) \\ R_N + j\left(X_{BA} + X_{BN} \right) & R_B + R_N + j\left(X_{BB} + X_{BN} \right) & R_N + j\left(X_{BC} + X_{BN} \right) \\ R_N + j\left(X_{CA} + X_{CN} \right) & R_N + j\left(X_{CB} + X_{CN} \right) & R_C + R_N + j\left(X_{CC} + X_{CN} \right) \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \end{aligned} \tag{14}$$

As an example of reading this set of equations, the voltage drop for phase A is:

$$\begin{aligned} &V_{AN,S} - V_{AN,R} = \\ &\left(R_A + R_N + j\left(X_{AA} + X_{AN} \right) \right) I_A + \left(R_N + j\left(X_{AB} + X_{AN} \right) \right) I_B + \left(R_N + j\left(X_{AC} + X_{AN} \right) \right) I_C \end{aligned} \tag{15}$$

A shorthand way of stating (14) is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \tag{16}$$

Equation (16) can be abbreviated further to:

$$\mathbf{V_{ABC,S}} - \mathbf{V_{ABC,R}} = \mathbf{Z_{ABC}} \cdot \mathbf{I_{ABC}} \tag{17}$$

Note in (14)-(17) that:

- Wye voltages that reference neutral are implied.
- Positive current is implied to be from the sending end to the receiving end.
- The voltage to remote ground, V_{NG}, is neither utilized nor calculated explicitly.
- In the abbreviated equation (16) the “N” reference for voltages is assumed and not explicitly stated. The N reference will be dropped in the rest of the paper.
- Note the implied definition of $\mathbf{Z_{ABC}}$:

$$\mathbf{Z_{ABC}} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \tag{18}$$

where:

$Z_{AA} = V_{AS}\text{-}V_{AR} / I_A$ $\quad Z_{AB} = V_{AS}\text{-}V_{AR} / I_B$ $\quad Z_{AC} = V_{AS}\text{-}V_{AR} / I_C$

$Z_{BA} = V_{BS}\text{-}V_{BR} / I_A$ $\quad Z_{BB} = V_{BS}\text{-}V_{BR} / I_B$ $\quad Z_{BC} = V_{BS}\text{-}V_{BR} / I_C$

$Z_{CA} = V_{CS}\text{-}V_{CR} / I_A$ $\quad Z_{CB} = V_{CS}\text{-}V_{CR} / I_B$ $\quad Z_{CC} = V_{CS}\text{-}V_{CR} / I_C$

Restating the matter of self and mutual impedances, the terms Z_{AA}, Z_{BB}, and Z_{CC} are referred to as self impedances (e.g., Z_{AA} is a measure that relates phase A voltage and phase A current), and all other inter-phase quantities are referred to as mutual impedances (e.g., Z_{AB} is a measure that interrelates phase A voltage and phase B current).

Figure 3 is a lumped parameter approach to viewing all the impedances described in (14), as well as elsewhere above.

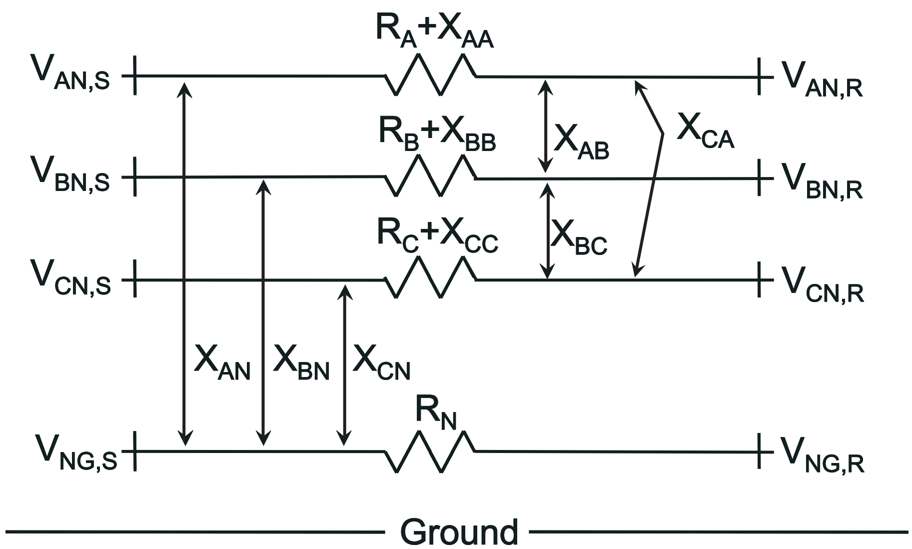

Figure 3 - Lumped Parameter 3 Phase Impedance Network

As a side note, while the only convenient way to show the inductance of an electric circuit is via a lumped parameter on a conductor, this method can be technically a little bit misleading, possibly giving erroneous perceptions of the nature of inductive impedances. The lumped inductance diagram leaves the impression that a wire, all by itself, has inductance. Actually, inductance is not defined until the conductive route back to the source is defined, which in turn defines the area of the loop and the flux inside the loop. Inductance must be eventually defined by a conductive loop and equation (1).

Neutral versus Ground

In this paper, the term “neutral” is utilized for voltage reference, and faults are “to neutral” rather than “to ground.” The term neutral is used rather than ground to help differentiate a current carrying neutral from the remote equipotential earth ground. Refer to figure 2. Ground in this figure carries no current and the voltage drop is in the neutral conductor. All voltages are generated against the local neutral, and V_{NG} at the sending and receiving ends will be different if any neutral current flows.

There is a small problem with the concept of a fault to ground. As soon as current is carried in the ground, there is a voltage drop in the ground resistance. Hence, ground potential is no longer uniform, and your ground reference has moved away from you. The ground reference is always just out of reach whenever current is flowing, so the concept of ground fault resistance is sometimes introduced. An alternate concept is to say that a fault is not to ground but to the local neutral as seen in figure 2 and to say that the true ground never sees any current flow.

Review of Symmetrical Components Concepts

A basic tenet of symmetrical component analysis of a power system is that any set of ABC phase domain voltages or currents (called the ABC domain in the balance of the paper) can be restated as a sum of three balanced symmetrical component domain voltages or currents (called the 012 domain in the balance of this paper). An example set of equivalent ABC and 012 domain voltages is shown in figure 4.

The terms “symmetrical coordinates” (Fortescue’s original 1918 term), “sequence components,” “symmetrical components,” “sequence component circuit,” and “symmetrical component network” are very similar and used almost interchangeably in papers and texts on this topic. In this paper, the term “symmetrical components” generally refers to the entire concept of this type of analysis, and a “sequence component” generally refers to the individual elements (positive, negative, and zero sequence) within the larger symmetrical component analysis approach. Also, the term “symmetrical component network” will be used rather than “sequence component circuit.”

The equations that correlate ABC and 012 domain voltages and currents (showing both is a little redundant since they are parallel equations) are:

$$\begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} \qquad \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \qquad \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \tag{20}$$

where the definition of the “a” and “a^2” are simply phase shifts of 120 and 240 degrees respectively. In symmetrical component analysis, at least as it has been applied almost universally, the positive sequence component is assumed to have an ABC phase rotation, and phase A is the reference for the sequence quantities. For example, in figure 4, you will see that the phase A component of V_0 is shown at 80^0, the phase A component of V_1 is shown at 20^0, and the phase A component of V_2 is shown at 30^0. These are the angles of the respective sequence quantities V_0, V_1, and V_2.

Some texts and applications use the terms V_{A0}, V_{A1}, V_{A2}, V_{B0}, V_{B1}, V_{B2}, V_{C0}, V_{C1}, and V_{C2} when specifying sequence component quantities as measured on the various phases and, therefore, avoid explicitly including the “a” or “a^2” factor in the text or equation. See figure 4 for an example use of this notation. In this paper, the “a” and “a^2” term will generally be explicitly stated rather than indicating phase shifts via the use of subscripts such as these, and whenever V_0, V_1, and V_2 are mentioned, V_{A0}, V_{A1}, and V_{A2} will be implied.

It should also be noted that there is an uncommonly used version of the symmetrical component equations, called the “power invariant” equations, that can be recognized by a 1/sqrt(3) factor in front of each conversion matrix in (19) and (20) above. This version of symmetrical component theory will not be used herein.

The set of equations in (19) and (20) is frequently shown in an abbreviated shorthand method. First, the conversion matrices are abbreviated to **A** and $\mathbf{A}^{-1}$:

$$\mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{vmatrix} \tag{21}$$

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \tag{22}$$

This allows (19) and (20) to be stated as:

$$\begin{matrix} \mathbf{V_{ABC}} = \mathbf{A} \cdot \mathbf{V_{012}} & \quad & \mathbf{V_{012}} = \mathbf{A}^{-1} \cdot \mathbf{V_{ABC}} \\ \mathbf{I_{ABC}} = \mathbf{A} \cdot \mathbf{I_{012}} & \quad & \mathbf{I_{012}} = \mathbf{A}^{-1} \cdot \mathbf{I_{ABC}} \end{matrix} \tag{23}$$

As we proceed, it will be important to know that:

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

As an example, a case of the conversion between ABC and 012 domain voltages is seen in figure 4. The math for figure 4 is:

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} 2.66\angle 46 \\ 0.89\angle 138 \\ 1.17\angle 118 \end{bmatrix} = \begin{bmatrix} 1.2\angle 80 \\ 1.0\angle 20 \\ 0.8\angle 30 \end{bmatrix}$$

$$\begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}\begin{bmatrix} 1.2\angle 80 \\ 1.0\angle 20 \\ 0.8\angle 30 \end{bmatrix} = \begin{bmatrix} 2.66\angle 46 \\ 0.89\angle 138 \\ 1.17\angle 118 \end{bmatrix}$$

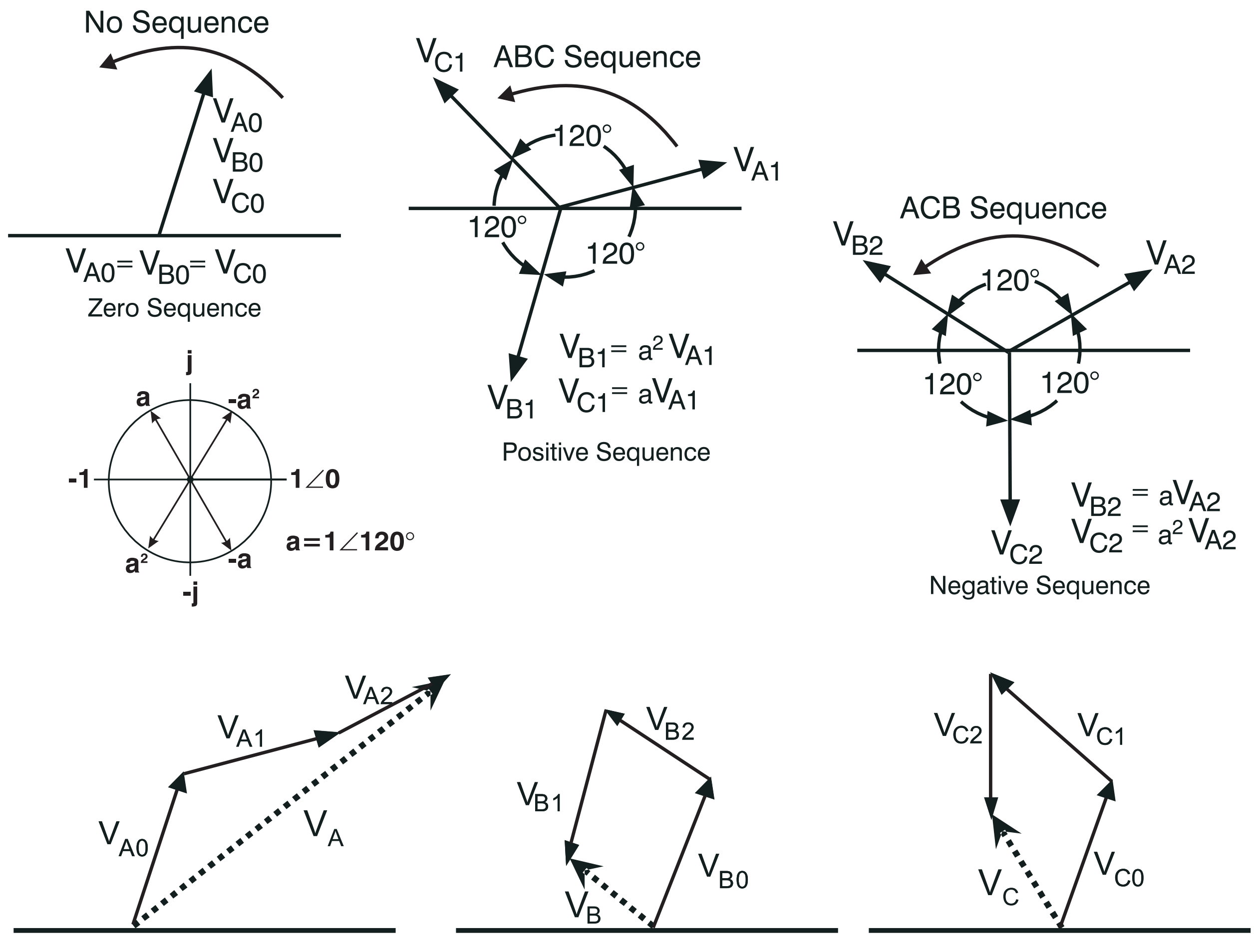

Figure 4 - Example Sequence Component Phasors

ABC/012 Conversion of the Voltage Drop Equation

To progress to the derivation of the symmetrical component networks and perform general fault analysis, we need to be able to express the ABC domain voltage drop equation in 012 domain terms. Below we will convert the entire voltage drop equation from the ABC domain to the 012 domain and, conversely, from the 012 domain back to the ABC domain. The equations for conversion of voltages and currents to and from the ABC and 012 domains were given above, (19) and (20). To complete the conversion, we need to be able to convert the impedance portion of the voltage drop equation.

Recall the ABC domain voltage drop equation (16):

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}$$

One should recall that in matrix mathematics, similar to algebraic manipulation, if we multiply all elements on both sides of the equation by the same matrix, we will not have changed the equality of the two sides of the equation. In this case we will multiply across by $\mathbf{A}^{-1}$. Further, if we multiply by $\mathbf{A \cdot A^{-1}}$ (24) we have effectively only multiplied by 1, so the equality below is a valid modification of (16):

$$\mathbf{A}^{-1}\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \mathbf{A}^{-1}\begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix}\mathbf{A} \cdot \mathbf{A}^{-1}\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \tag{25}$$

This equation, when multiplied out, will become the 012 domain voltage drop equation. The reader should already recognize the conversion of $\mathbf{V_{ABC}}$ and $\mathbf{I_{ABC}}$ to $\mathbf{V_{012}}$ and $\mathbf{I_{012}}$ from the previous material. The form of the resultant equation will be:

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \tag{26}$$

which can be abbreviated further to:

$$\mathbf{V}_{012,\mathbf{S}} - \mathbf{V}_{012,\mathbf{R}} = \mathbf{Z}_{012} \cdot \mathbf{I}_{012} \tag{27}$$

In (26) and (27), note the implied definition of $\mathbf{Z_{012}}$:

$$\mathbf{Z}_{012} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix} \tag{28}$$

where:

$Z_{00} \Rightarrow (V_{0,S} - V_{0,R}) / I_0$ $\quad$ $Z_{01} \Rightarrow (V_{0,S} - V_{0,R}) / I_1$ $\quad$ $Z_{02} \Rightarrow (V_{0,S} - V_{0,R}) / I_2$
$Z_{10} \Rightarrow (V_{1,S} - V_{1,R}) / I_0$ $\quad$ $Z_{11} \Rightarrow (V_{1,S} - V_{1,R}) / I_1$ $\quad$ $Z_{12} \Rightarrow (V_{1,S} - V_{1,R}) / I_2$
$Z_{20} \Rightarrow (V_{2,S} - V_{2,R}) / I_0$ $\quad$ $Z_{21} \Rightarrow (V_{2,S} - V_{2,R}) / I_1$ $\quad$ $Z_{22} \Rightarrow (V_{2,S} - V_{2,R}) / I_2$

The equation to convert from $\mathbf{Z_{ABC}}$ to $\mathbf{Z_{012}}$ is the $\mathbf{A^{-1} Z_{ABC} A}$ portion of (25):

$$\mathbf{Z_{012}} = \mathbf{A}^{-1}\begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix}\mathbf{A} \tag{26}$$

The set of equations implied by the matrix math of (26) is:

$$
\begin{aligned}
Z_{00} &= \left(Z_{AA} + Z_{BA} + Z_{CA} + Z_{AB} + Z_{BB} + Z_{CB} + Z_{AC} + Z_{BC} + Z_{CC} \right) / 3 \\
Z_{01} &= \left(Z_{AA} + Z_{BA} + Z_{CA} + a^2 Z_{AB} + a^2 Z_{BB} + a^2 Z_{CB} + aZ_{AC} + aZ_{BC} + aZ_{CC} \right) / 3 \\
Z_{02} &= \left(Z_{AA} + Z_{BA} + Z_{CA} + aZ_{AB} + aZ_{BB} + aZ_{CB} + a^2 Z_{AC} + a^2 Z_{BC} + a^2 Z_{CC} \right) / 3 \\
Z_{10} &= \left(Z_{AA} + aZ_{BA} + a^2 Z_{CA} + Z_{AB} + aZ_{BB} + a^2 Z_{CB} + Z_{AC} + aZ_{BC} + a^2 Z_{CC} \right) / 3 \\
Z_{11} &= \left(Z_{AA} + aZ_{BA} + a^2 Z_{CA} + a^2 Z_{AB} + Z_{BB} + aZ_{CB} + aZ_{AC} + a^2 Z_{BC} + Z_{CC} \right) / 3 \\
Z_{12} &= \left(Z_{AA} + aZ_{BA} + a^2 Z_{CA} + aZ_{AB} + a^2 Z_{BB} + Z_{CB} + a^2 Z_{AC} + Z_{BC} + aZ_{CC} \right) / 3 \\
Z_{20} &= \left(Z_{AA} + a^2 Z_{BA} + aZ_{CA} + Z_{AB} + a^2 Z_{BB} + aZ_{CB} + Z_{AC} + a^2 Z_{BC} + aZ_{CC} \right) / 3 \\
Z_{21} &= \left(Z_{AA} + a^2 Z_{BA} + aZ_{CA} + a^2 Z_{AB} + aZ_{BB} + Z_{CB} + aZ_{AC} + Z_{BC} + a^2 Z_{CC} \right) / 3 \\
Z_{22} &= \left(Z_{AA} + a^2 Z_{BA} + aZ_{CA} + aZ_{AB} + Z_{BB} + a^2 Z_{CB} + a^2 Z_{AC} + aZ_{BC} + Z_{CC} \right) / 3
\end{aligned}
\tag{30}
$$

There is an inverse process to convert from the 012 domain voltage drop equation to the ABC domain. We simply multiply the 012 domain voltage drop equation (26) by **A**, and include $\mathbf{A}^{-1} \cdot \mathbf{A}$ as indicated below:

$$
\mathbf{A}\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \mathbf{A}\begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \mathbf{A}\begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix} \mathbf{A}^{-1} \cdot \mathbf{A}\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \tag{31}
$$

which returns us to the ABC domain voltage drop equation (16):

$$
\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}
$$

The equation to convert from $\mathbf{Z_{012}}$ to $\mathbf{Z_{ABC}}$ is the $\mathbf{A\,Z_{012}A^{-1}}$ portion of (31):

$$
\mathbf{Z_{ABC}} = \mathbf{A}\begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix} \mathbf{A}^{-1} \tag{32}
$$

The set of equations implied by the matrix math in (32) is:

$$
\begin{aligned}
Z_{AA} &= \left(Z_{00} + Z_{10} + Z_{20} + Z_{01} + Z_{11} + Z_{21} + Z_{02} + Z_{12} + Z_{22} \right) / 3 \\
Z_{AB} &= \left(Z_{00} + Z_{10} + Z_{20} + aZ_{01} + aZ_{11} + aZ_{21} + a^2 Z_{02} + a^2 Z_{12} + a^2 Z_{22} \right) / 3 \\
Z_{AC} &= \left(Z_{00} + Z_{10} + Z_{20} + a^2 Z_{01} + a^2 Z_{11} + a^2 Z_{21} + aZ_{02} + aZ_{12} + aZ_{22} \right) / 3 \\
Z_{BA} &= \left(Z_{00} + a^2 Z_{10} + aZ_{20} + Z_{01} + a^2 Z_{11} + aZ_{21} + Z_{02} + a^2 Z_{12} + aZ_{22} \right) / 3 \\
Z_{BB} &= \left(Z_{00} + a^2 Z_{10} + aZ_{20} + aZ_{01} + Z_{11} + a^2 Z_{21} + a^2 Z_{02} + aZ_{12} + Z_{22} \right) / 3 \\
Z_{BC} &= \left(Z_{00} + a^2 Z_{10} + aZ_{20} + a^2 Z_{01} + aZ_{11} + Z_{21} + aZ_{02} + Z_{12} + a^2 Z_{22} \right) / 3 \\
Z_{CA} &= \left(Z_{00} + aZ_{10} + a^2 Z_{20} + Z_{01} + aZ_{11} + a^2 Z_{21} + Z_{02} + aZ_{12} + a^2 Z_{22} \right) / 3 \\
Z_{CB} &= \left(Z_{00} + aZ_{10} + a^2 Z_{20} + aZ_{01} + a^2 Z_{11} + Z_{21} + a^2 Z_{02} + Z_{12} + aZ_{22} \right) / 3 \\
Z_{CC} &= \left(Z_{00} + aZ_{10} + a^2 Z_{20} + a^2 Z_{01} + Z_{11} + aZ_{21} + aZ_{02} + a^2 Z_{12} + Z_{22} \right) / 3
\end{aligned}
\qquad (33)
$$

Impedance Matrices with High Symmetry

The impedance matrix in the 012 domain and the associated voltage drop equation becomes relatively simple for two certain cases of symmetrical system impedances:

1. Symmetrical Impedance Passive Elements
   In such elements, each phase's self impedance is the same uniform value, and phase to phase mutual coupling is uniform. Examples are transposed and symmetrically spaced transmission lines, and through impedances of transformers.
   Result to seen: $Z_{11} = Z_{22}$, and $Z_{01} = Z_{02} = Z_{10} = Z_{12} = Z_{20} = Z_{21} = 0$
2. Electric Machines
   In machines, each phase's self impedance is the same, but phase to phase mutual coupling involves differing "forward" and "reverse" mutual coupling.
   Result to seen: $Z_{11} \neq Z_{22}$, and $Z_{01} = Z_{02} = Z_{10} = Z_{12} = Z_{20} = Z_{21} = 0$

In these two cases, the symmetry of the $\mathbf{Z_{ABC}}$ matrix results in a $\mathbf{Z_{012}}$ matrix with only diagonal elements. This means that in the 012 domain voltage drop equation, there will be no mutual coupling in voltage drops; e.g., V_0 is due only to I_0 flow and is not directly due to I_1 and I_2 flow.

It is not always apparent that the commonly used terms Z_0, Z_1, and Z_2 by implication tend to refer to this ideal balanced impedance state identified by conditions 1 and 2 above. In this case, Z_0, Z_1, and Z_2 refer to Z_{00}, Z_{11}, and Z_{22} where the double subscript notation has been dropped for simplicity and where the mutual terms in $\mathbf{Z_{012}}$ have gone to 0. This matter will become clearer in (38), (39), and (46) below.

Case 1 - Symmetrical Passive Elements

In transmission lines, especially when phases are symmetrically spaced and phases are regularly transposed, it is frequently justified to assume the following symmetries in the $\mathbf{Z_{ABC}}$ impedance network:

$$
\begin{aligned}
&X_{SELF}\left(X_S\right)=X_{AA}=X_{BB}=X_{CC}\\
&X_{MUTUAL}\left(X_M\right)=X_{AB}=X_{AC}=X_{BA}=X_{BC}=X_{CA}=X_{CB}\\
&X_{NEUTRAL}\left(X_N\right)=X_{AN}=X_{BN}=X_{CN}\\
&R_{PHASE}\left(R_P\right)=R_A=R_B=R_C
\end{aligned} \tag{34}
$$

Substituting these impedances into (14) gives the following definitions of Z_S and Z_M and the various elements of $\mathbf{Z_{ABC}}$:

$$
\begin{aligned}
Z_S &= Z_{AA}=Z_{BB}=Z_{CC}\\
&=R_P+R_N+j\left(X_S+X_N\right)\\
Z_M &= Z_{AB}=Z_{AC}=Z_{BA}=Z_{BC}=Z_{CA}=Z_{CB}\\
&=R_N+j\left(X_M+X_N\right)
\end{aligned} \tag{35}
$$

Equations (34) and (35) give the $\mathbf{Z_{ABC}}$ matrix the form:

$$
\mathbf{Z_{ABC}}=\begin{bmatrix} Z_S & Z_M & Z_M\\ Z_M & Z_S & Z_M\\ Z_M & Z_M & Z_S \end{bmatrix} \tag{36}
$$

When we convert $\mathbf{Z_{ABC}}$ to $\mathbf{Z_{012}}$, we obtain the relatively simple diagonal matrix:

$$
\mathbf{Z_{012}}=\mathbf{A}^{-1}\begin{bmatrix} Z_S & Z_M & Z_M\\ Z_M & Z_S & Z_M\\ Z_S & Z_M & Z_S \end{bmatrix}\mathbf{A}=\begin{bmatrix} Z_S+2Z_M & 0 & 0\\ 0 & Z_S-Z_M & 0\\ 0 & 0 & Z_S-Z_M \end{bmatrix} \tag{37}
$$

Hence, the 012 domain voltage drop equation becomes:

$$
\begin{bmatrix} V_{0,S}\\ V_{1,S}\\ V_{2,S} \end{bmatrix}-\begin{bmatrix} V_{0,R}\\ V_{1,R}\\ V_{2,R} \end{bmatrix}=\begin{bmatrix} Z_S+2Z_M & 0 & 0\\ 0 & Z_S-Z_M & 0\\ 0 & 0 & Z_S-Z_M \end{bmatrix}\begin{bmatrix} I_0\\ I_1\\ I_2 \end{bmatrix} \tag{38}
$$

Equation (38) is typically stated as:

$$
\begin{bmatrix} V_{0,S}\\ V_{1,S}\\ V_{2,S} \end{bmatrix}-\begin{bmatrix} V_{0,R}\\ V_{1,R}\\ V_{2,R} \end{bmatrix}=\begin{bmatrix} Z_0 & 0 & 0\\ 0 & Z_1 & 0\\ 0 & 0 & Z_2 \end{bmatrix}\begin{bmatrix} I_0\\ I_1\\ I_2 \end{bmatrix} \tag{39}
$$

where:

$$
\begin{aligned}
Z_{00}=Z_0 &= Z_S+2Z_M\\
&=\left(R_P+3R_N\right)+j\left(X_S+2X_M+3X_N\right)\\
Z_{11}=Z_{22}=Z_1=Z_2 &= Z_S-Z_M\\
&=R_P+j\left(X_S-X_M\right)
\end{aligned} \tag{40}
$$

Also note that via some algebraic manipulation of (40):

$$
\begin{aligned}
Z_S &= \left(Z_0 + 2Z_1\right)/3 \\
Z_M &= \left(Z_0 - Z_1\right)/3
\end{aligned}
\tag{41}
$$

Equation (41) also could be derived, and some extra verification or understanding or the Z_S and Z_M quantities could be obtained, by application of (33). Start by accepting the classical assumption that $\mathbf{Z_{012}}$ is of the form given by (39) where all mutual terms are 0, then assume $Z_1 = Z_2$, and then apply these assumptions to (33) to find $\mathbf{Z_{ABC}}$. One will find $\mathbf{Z_{ABC}}$ has the form given in (36) and (41).

Case 2: Electric Machines

We know that, in some cases, $Z_1 \neq Z_2$, which is most commonly seen in electric machine impedances. A direct result of $Z_1 \neq Z_2$ (with mutual terms in $\mathbf{Z_{012}} = 0$), is that two different X_M quantities (X_{M+} and X_{M-}) are needed to represent the mutual impedances in the $\mathbf{Z_{ABC}}$ matrix (developed further in (48) below). So, when $Z_1 \neq Z_2$, the following impedance symmetry exists and represents machine impedance in the ABC domain:

$$
\begin{aligned}
&X_{SELF}\left(X_S\right) = X_{AA} = X_{BB} = X_{CC} \\
&X_{MUTUAL,POSITIVE}\left(X_{M+}\right) = X_{AB} = X_{BC} = X_{CA} \\
&X_{MUTUAL,NEGATIVE}\left(X_{M-}\right) = X_{BA} = X_{CB} = X_{AC} \\
&X_{NEUTRAL}\left(X_N\right) = X_{AN} = X_{BN} = X_{CN} \\
&R_{PHASE}\left(R_P\right) = R_A = R_B = R_C
\end{aligned}
\tag{42}
$$

Substituting the impedances of (42) into (14) gives the following definitions of Z_S, Z_{M+}, and Z_{M-} and the various elements of $\mathbf{Z_{ABC}}$:

$$
\begin{aligned}
Z_S &= R_P + R_N + j\left(X_S + X_N\right) \\
Z_{M+} &= R_N + j\left(X_{M+} + X_N\right) \\
Z_{M-} &= R_N + j\left(X_{M-} + X_N\right)
\end{aligned}
\tag{43}
$$

Equations (42) and (43) give the $\mathbf{Z_{ABC}}$ matrix the form:

$$
\mathbf{Z_{ABC}} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix}
\tag{44}
$$

The effect of differing Z_{M+} and Z_{M-} is that, in a rotating machine, current in phase B has a different mutual coupling to the forward looking phase A (Z_{M+}) than the backward looking phase C (Z_{M-}). When we convert $\mathbf{Z_{ABC}}$ to $\mathbf{Z_{012}}$ we obtain a result similar to (37):

$$\mathbf{Z_{012}} = \mathbf{A}^{-1}\begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix}\mathbf{A}$$
$$= \begin{bmatrix} Z_S + Z_{M+} + Z_{M-} & 0 & 0 \\ 0 & Z_S + a^2 Z_{M+} + aZ_{M-} & 0 \\ 0 & 0 & Z_S + aZ_{M+} + a^2 Z_{M-} \end{bmatrix} \quad (45)$$

Hence, the 012 domain voltage drop equation becomes:

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_S + Z_{M+} + Z_{M-} & 0 & 0 \\ 0 & Z_S + a^2 Z_{M+} + aZ_{M-} & 0 \\ 0 & 0 & Z_S + aZ_{M+} + a^2 Z_{M-} \end{bmatrix}\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \quad (46)$$

Again, this equation is typically stated as (39):

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix}\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix}$$

but in this case:

$$\begin{aligned} Z_{00} = Z_0 &= Z_S + Z_{M+} + Z_{M-} \\ &= R_P + 3R_N + j\left(X_S + X_{M+} + X_{M-} + 3X_N\right) \\ Z_{11} = Z_1 &= Z_S + a^2 Z_{M+} + aZ_{M-} \\ &= R_P + j\left(X_S + a^2 X_{M+} + aX_{M-}\right) \\ Z_{22} = Z_2 &= Z_S + aZ_{M+} + a^2 Z_{M-} \\ &= R_P + j\left(X_S + aX_{M+} + a^2 X_{M-}\right) \end{aligned} \quad (47)$$

If one needs some extra verification or understanding of the Z_S, Z_{M+}, and Z_{M-} quantities, one can work in a reverse direction by accepting the classical assumption that $\mathbf{Z_{012}}$ is of the form in (39) where all mutual terms are 0, accepting that $Z_1 \neq Z_2$, and then applying these assumptions to find $\mathbf{Z_{ABC}}$ via (33). One will find $\mathbf{Z_{ABC}}$ has the form given in (44) where:

$$\begin{aligned} Z_S &= \left(Z_0 + Z_1 + Z_2\right)/3 \\ Z_{M+} &= \left(Z_0 + aZ_1 + a^2 Z_2\right)/3 \\ Z_{M-} &= \left(Z_0 + a^2 Z_1 + aZ_2\right)/3 \end{aligned} \quad (48)$$

If we substitute (47) into (48), we will obtain (43). Also note in (47) that if $Z_{M+} = Z_{M-}$, then $Z_1 = Z_2$ and (47) reduces to (40). Conversely, note in (48) that if $Z_1 = Z_2$, then $Z_{M+} = Z_{M-}$, and (48) reduces to (41).

Phase Impedance Unbalances Causing Mutual Coupling in the Z_{012} Matrix

In the two cases above, the symmetry of the $\mathbf{Z_{ABC}}$ matrix results in a $\mathbf{Z_{012}}$ matrix with only diagonal elements. This means in each case that, in the 012 domain voltage drop equation, $V_0 = Z_0 I_0$, $V_1 = Z_1 I_1$, and $V_2 = Z_2 I_2$. More to the point, there is no case of cross coupling, where one sequence component current directly feeds into a different sequence voltage; e.g., there is no case of $V_0 = Z_{01} I_1$. This will be a valuable concept when developing symmetrical component networks, below.

However, consider how the Z_{012} matrix would appear if the self impedance of phase A was increased by ΔZ_{AA} (e.g., assume an increase of $\Delta R_A + \Delta X_{AA}$ in (30)). Inspection of (30) will show that every element in the Z_{012} matrix will increase by $\Delta Z_{AA}/3$, including all mutual elements. There will be a similar effect if any phase to phase mutual impedance differs from the assumptions made in (34) and (42). Hence, the diagonal $\mathbf{Z_{012}}$ matrix of (39) occurs only if the assumptions of (34) and (42) are true. This may be a questionable matter at times.

Symmetrical Component Networks

With this groundwork on symmetrical component analysis, particularly the nature of the basic three phase ABC and 012 domain voltage drop equation, let us proceed with development of the symmetrical component networks and the general equations for short circuits and open circuits. The process to develop these networks will follow these steps:

1. State the voltage drop equation in ABC domain for the known system values during the fault. Also state the equation in the 012 domain.
2. Solve for the ABC and the 012 domain voltages and currents.
3. Devise an electric circuit (i.e., an symmetrical component network) consisting of 012 domain voltages and impedances that has the same solution for currents as the mathematically derived solution for 012 domain currents found in step 2.

Restating steps 1-3 in, perhaps, a more understandable fashion: To find a symmetrical component network that represents a given fault, one first starts by defining the problem in the ABC domain. Then, examining the physics of the event and working in both the ABC and 012 domains, one works to find equations that state system currents in the 012 domain in terms of 012 voltages and impedances. Some analysis of currents using the ABC domain voltage drop equation will assist in developing the 012 domain equations for current. Once the 012 domain equations of current are found, one goes back to devise an electric circuit (i.e., a symmetrical component network) that has the same mathematics for current as the derived 012 domain current equations. Note that we find the circuit last, and one is finding circuits to mimic the 012 domain equations. One is not creating equations to match intuitively seen physical circuits.

There is a symmetrical component network that can represent any fault, but in many cases the networks are intricate and analysis is not intuitive. Few ever use them in actual practice. There are several reasons:

- Some networks may require imaginary and non-intuitive phase shifting transformers to show the "a" and "a^2" factors that appear in the equations. For example, see the network required to model the solution in the phase B to neutral analysis to follow, as well as the equations seen in Appendix A.
- Finding the circuits is a complex process, and people tend to distrust working with the results if they do not understand the origin of the equations.
- There is not much need to be able to calculate a fault of every type on every possible phase. The subset of easy-to-work-with networks is sufficient for most applications.

The analysis process below will vary a bit from the process found in some of the more common texts and papers on symmetrical component analysis:

- The analysis below does more development in the ABC domain than is common. One rationale is to help the reader become more comfortable with using the ABC domain equations for fault current and voltage drop analysis.
- The analysis below works with a $\mathbf{Z_{ABC}}$ matrix where Z_M is separated into Z_{M+} and Z_{M-}, which in turn is a result of allowing $Z_1 \neq Z_2$ in the analysis.

Rather then develop a symmetrical component network for every fault type, the equations for each fault type on every phase will be given in Appendix A, and only some mainstream “easy” networks that are commonly used and do not have phase shifting transformers will be presented. However, the phase B to neutral fault will be developed as a learning tool.

Fault Impedance

The most common approach to including fault impedance is to insert a representative extra impedance at specified locations in the symmetrical component networks. The approach gives the correct results but is one step removed from a physical understanding of what is occurring. If one wishes to return to the basics of the physical world, it can be seen that fault impedance is actually creating a modification of the $\mathbf{Z_{ABC}}$ impedance matrix in (14), which, in turn, affects the sequence impedances calculated in (40) and (47). Rather than inserting possibly non-intuitive impedances in the symmetrical component networks, it may be more intuitive, less error prone, and in some cases more accurate to see how the assumed fault impedance modifies $\mathbf{Z_{ABC}}$ as seen in (14) and subsequently see how this affects Z_0, Z_1, and Z_2 as seen in (40) and (47).

The simple but common example is when fault impedance is considered a resistance that is added to the neutral or phase leg of the circuit:

- Increased Neutral Resistance:
  Inspection of (40) and (47) shows that fault resistance $R_{N,F}$ added in the neutral causes an increase in Z_0 by a factor of $3R_{N,F}$ and has no effect on Z_1 or Z_2.
- Increased Phase Resistance:
  To maintain symmetry in the $\mathbf{Z_{ABC}}$ matrix and keep the impedance symmetries in (34) and (42) valid and, hence, keep the diagonal simplicity of $\mathbf{Z_{012}}$ true and valid, we need to assume that the impedance shows up in every phase. In a phase-phase fault, we do this by assuming half the fault impedance is added to R_P on every phase. In theory, the unfaulted phase will see the same increase in resistance, but it will not affect the results, since no current flows in the unfaulted phase. The net effect according to (40) and (47) is to increase Z_0, Z_1, and Z_2 each by one half of $R_{N,F}$.

Shunt Faults

Shunt faults to neutral are modeled by specifying that in the voltage drop equation (16), V_R on one or more phases is 0 (due to a phase to neutral fault), that a yet to be determined current I_F flows on the faulted phases, and that 0 current flows on the unfaulted phases. For a phase to phase fault, we assume two voltages at the receiving end are identical relative to neutral. On the unfaulted phases, we assume that there is a high impedance or open circuit at the receiving end on those phases (i.e., there is no load flow). We will start with the unbalanced phase to neutral and phase to phase faults and discuss the simple three phase fault last.

Phase A to Neutral Fault

The ABC domain voltage drop equation for a phase A to neutral fault at the receiving end is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} I_F \\ 0 \\ 0 \end{bmatrix} \tag{49}$$

Faulted System Conditions: Relative to the local neutral, the received voltage on phase A is 0 (due to the phase to neutral fault). The fault current I_F on phase A is unknown. The current on the unfaulted phases B and C is 0. The received voltage on the unfaulted phases B and C is unknown. The sending end voltage $\mathbf{V_{ABC,S}}$ is known.

Stating (49) in the 012 domain, using (25), we obtain:

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix} \begin{bmatrix} I_F/3 \\ I_F/3 \\ I_F/3 \end{bmatrix} \tag{50}$$

From an inspection of (19), since $V_{A,R} = 0$, we know that:

$$V_{A,R} = V_{0,R} + V_{1,R} + V_{2,R} = 0 \tag{51}$$

Between (50) and (51) we have quite enough to proceed in developing a solution for I_F, find $\mathbf{I_{012}}$, and subsequently develop the appropriate symmetrical component network for the equations. Basically, one should be able to see that one has the tools to solve for I_F in terms of the known data Z_0, Z_1, and Z_2, and subsequently $V_{0,S}$, $V_{1,S}$, and $V_{2,S}$. In most resources on this topic, the process of developing the symmetrical component network for this fault condition proceeds from equations along the lines of (50) and (51). However, let us take a different approach to learn the usefulness of ABC domain voltage drop equation, and let us fill in the blanks on some of the algebra that is involved.

The top line in (49) is enough for us to determine I_F. We can easily see the resulting equation:

$$V_{A,S} = Z_S I_F \tag{52}$$

From (48) we know that:

$$Z_S = \left(Z_0 + Z_1 + Z_2\right)/3$$

So, we see that:

$$I_F = I_A = \frac{V_{A,S}}{Z_S} = \frac{3V_{A,S}}{Z_0 + Z_1 + Z_2} \tag{53}$$

Hence, the ABC domain voltage drop equation has given us a good quick start on finding system currents. Further, as seen in (20):

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}$$

Substituting (51) and $I_B = I_C = 0$ into the above equation tells us that:

$$I_0 = I_1 = I_2 = \frac{V_{A,S}}{Z_0 + Z_1 + Z_2} \quad (54)$$

Equation (54) indicates that the symmetrical component network required for modeling the fault will need to be a series impedances circuit since only in a series circuit are all currents the same. Further, an inspection of (19) tells us that:

$$V_{A,S} = V_{0,S} + V_{1,S} + V_{2,S} \quad (55)$$

Substituting (55) into (54) gives us:

$$I_0 = I_1 = I_2 = \frac{V_{0,S} + V_{1,S} + V_{2,S}}{Z_0 + Z_1 + Z_2} \quad (56)$$

In the typical power system we assume $V_{0,S}$ and $V_{2,S}$ is negligible and, if this is the case, (56) can be reduced to:

$$I_0 = I_1 = I_2 = \frac{V_{1,S}}{Z_0 + Z_1 + Z_2} \quad (57)$$

Now we can see that the equations derived above give the same answer as the symmetrical component network commonly given for a phase A to neutral fault, Figure 5. In figure 5, note $V_{012,R}$ is stated as $V_{012,F}$ to be consistent with the way these types of drawings are usually shown, where $V_{Received}$ is the fault location voltage. Note also the phasor diagrams, figure 6, showing how the 012 phasors add to 0 in phases B and C, and add to create the total fault current in phase A.

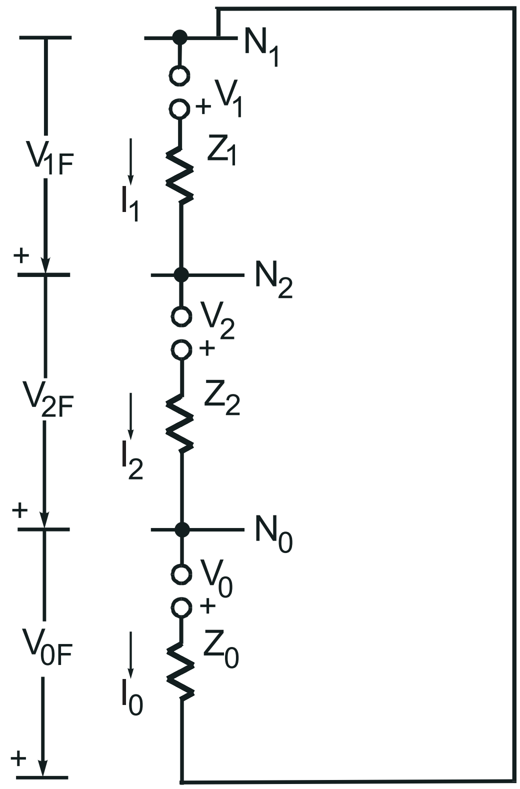

Figure 5 - Symmetrical Component Network - Phase A to Neutral Fault

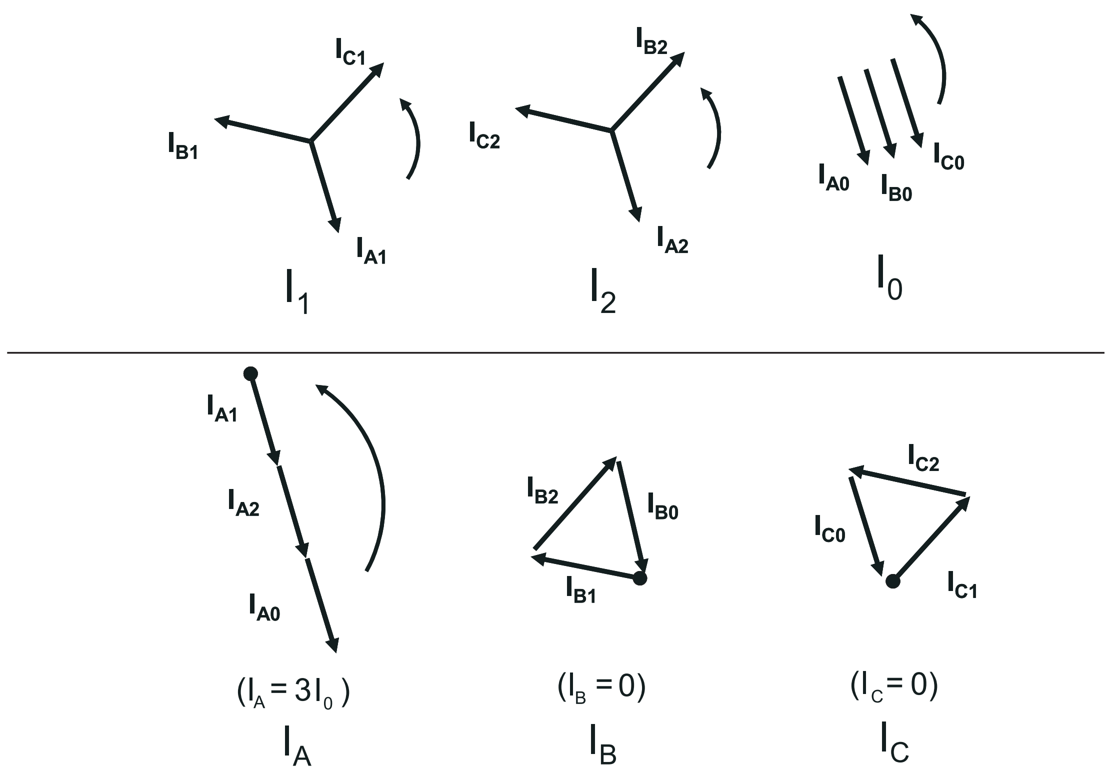

Figure 6 - Phasors for Phase A to Neutral Fault

Phase B to Neutral Fault

Only a subset of all the possible faults is commonly represented via symmetrical component networks. We will examine the phase B to neutral fault to help clarify the reason. We will see that the phase B to neutral fault does not lend itself to a circuit that is as clearly understood as figure 5.

The ABC domain voltage drop equation for a phase B to neutral fault at the receiving end is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ 0 \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} 0 \\ I_F \\ 0 \end{bmatrix} \tag{58}$$

Faulted System Conditions: The received voltage on phase B is 0, relative to the local neutral (due to the phase to neutral fault). The fault current I_F on phase B is unknown. The current on the unfaulted phases A and C is 0. The received voltage on the unfaulted phases A and C is unknown. The sending end voltage $\mathbf{V_{ABC,S}}$ is known.

Following the same thought process as for the phase A to neutral fault, the second line in (58) is enough for us to determine I_F. We can easily see the equation:

$$V_{B,S} = Z_S I_F \tag{59}$$

From (48) we know that:

$$Z_S = \left(Z_0 + Z_1 + Z_2 \right) / 3$$

Hence, we see that:

$$I_F = I_B = \frac{V_{B,S}}{Z_S} = \frac{3 \cdot V_{B,S}}{Z_0 + Z_1 + Z_2} \tag{60}$$

Further, as seen in (20):

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}$$

Substituting (60) and $I_A = I_B = 0$ into the above equation tells us that:

$$\begin{aligned} I_0 &= \frac{V_{B,S}}{Z_0 + Z_1 + Z_2} \\ I_1 &= \frac{aV_{B,S}}{Z_0 + Z_1 + Z_2} \\ I_2 &= \frac{a^2 V_{B,S}}{Z_0 + Z_1 + Z_2} \end{aligned} \tag{61}$$

Further, similar to (55), we can see from (19) and from the specified fault that:

$$
\begin{aligned}
V_{B,S} &= V_{0,S} + a^2 V_{1,S} + aV_{2,S} \\
V_{B,R} &= V_{0,R} + a^2 V_{1,R} + aV_{2,R} = 0
\end{aligned} \tag{62}
$$

Substituting the upper equation of (62) into (61) gives us:

$$
\begin{aligned}
I_0 &= \frac{V_{0,S} + a^2 V_{1,S} + aV_{2,S}}{Z_0 + Z_1 + Z_2} = a^2 I_1 \\
I_1 &= \frac{aV_{0,S} + V_{1,S} + a^2 V_{2,S}}{Z_0 + Z_1 + Z_2} \\
I_2 &= \frac{a^2 V_{0,S} + aV_{1,S} + V_{2,S}}{Z_0 + Z_1 + Z_2} = aI_1
\end{aligned} \tag{63}
$$

From the lower half of (62) and substituting in $V_{0,R}$, $V_{1,R}$, and $V_{2,R}$ as solved from (39), and multiplying across by "a," we will find the equation:

$$
a\left(V_{0,S} - Z_0 I_0\right) + \left(V_{1,S} - Z_1 I_1\right) + a^2\left(V_{2,S} - Z_2 I_2\right) = 0 \tag{64}
$$

This equation is represented by the network shown in figure 7, where again V_R is shown as V_F and the phasor diagrams for an example fault are shown in figure 8. The "a" constants in (64) and the ensuing phase shifting transformers have a detrimental effect on the ability to develop an easily understood symmetrical component network. Figure 7 is so user-unfriendly that very few would ever really try to use it. In actual practice, most just rely on the phase A to neutral fault information and ignore the phase B and C to neutral faults, but the circuit, phasor diagram, and the above analysis has some value. It is interesting to note the phase shifts in the fault current and the sequence currents that have occurred. The net phase fault current has shifted by the expected $V_{B,S}$ phase shift ("a^2"), but it might not have been immediately obvious that the sequence component quantities would have the phase shifts shown: I_1 has not seen any phase shift, yet the I_0 and I_2 phasors move in opposite directions.

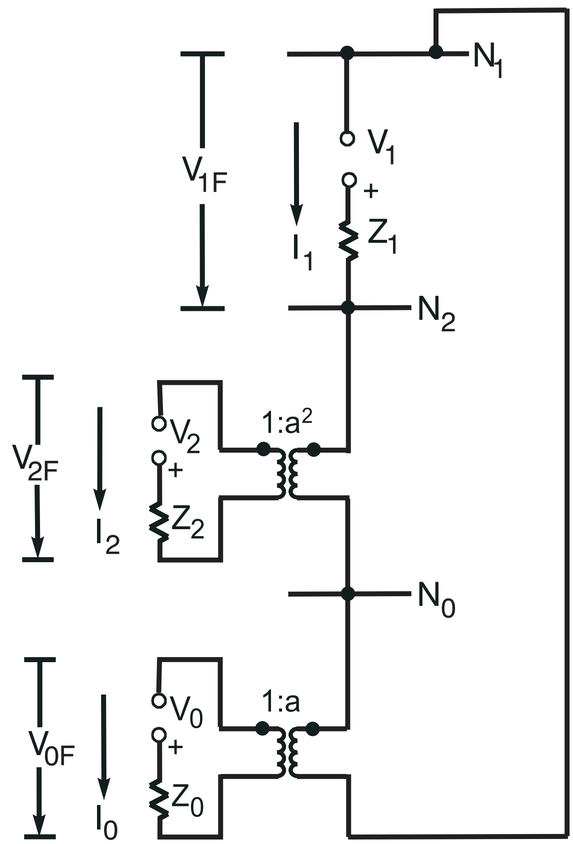

Figure 7 - Symmetrical Component Network - Phase B to Neutral Fault

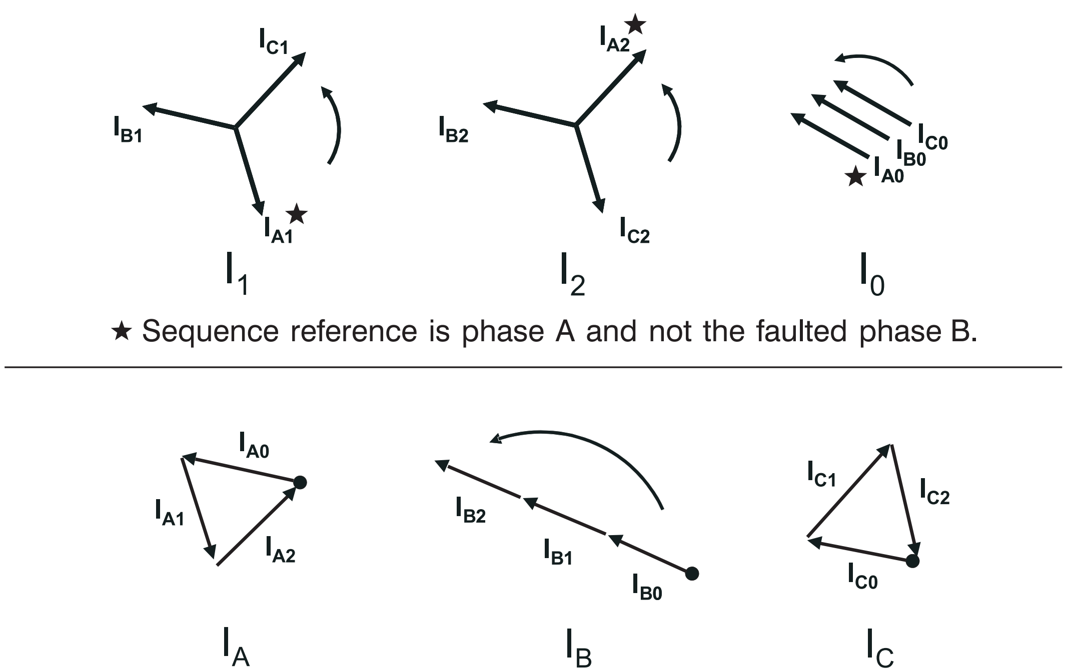

Figure 8 - Phasors for Phase B to Neutral Fault

Phase B to Phase C Fault

The ABC domain voltage drop equation for a phase B to phase C fault at the receiving end is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_F \\ V_F \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} 0 \\ I_F \\ -I_F \end{bmatrix} \tag{65}$$

Faulted System Conditions: The fault current I_F is the same on phase B and C but it is opposite in direction. The current on the unfaulted phase A is 0. The received voltage on phase B and C is the same, V_F (relative to the local neutral), but the specific voltage is unknown. The received voltage on phase A is unknown but $\mathbf{V_{ABC,S}}$ is known.

Stating (65) in the 012 domain, using (25), we obtain:

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix} \begin{bmatrix} 0 \\ jI_F/\sqrt{3} \\ -jI_F/\sqrt{3} \end{bmatrix} \tag{66}$$

From an inspection of (19), since we know that $V_{B,R}$=$V_{C,R}$, we can see that:

$$V_{0,R} + a^2V_{1,R} + aV_{2,R} = V_{0,S} + aV_{1,R} + V_{2,R} \tag{67}$$

One can tell from (67) that $V_{1,R}$ = $V_{2,R}$, which can be used to tell how to manipulate (66) to solve for I_F. Hence, one should see that, between (66) and (67), enough is known to proceed in developing an equation for I_F and subsequently find $\mathbf{I_{012}}$ and the appropriate symmetrical component network for the equations. In most resources on this topic, the process of developing a symmetrical component network for this fault condition proceeds from an equation set along the lines of (66) and (67). However, let us again take a different tactic, by starting with using the ABC domain voltage drop equation to define I_F, and let us fill in the blanks on some of the algebra that is involved.

Subtracting the bottom two lines in (65) gives us:

$$V_{B,S} - V_{C,S} = \left(2Z_S - Z_{M+} - Z_{M-}\right)I_F \tag{68}$$

From (48) we know that:

$$\begin{aligned} Z_S &= \left(Z_0 + Z_1 + Z_2\right)/3 \\ Z_{M+} &= \left(Z_0 + aZ_1 + a^2Z_2\right)/3 \\ Z_{M-} &= \left(Z_0 + a^2Z_1 + aZ_2\right)/3 \end{aligned}$$

Substituting the above equalities into (68) and solving for I_F gives us:

$$I_F = I_B = \frac{V_{B,S} - V_{C,S}}{Z_1 + Z_2} \tag{69}$$

Further, as seen before (20):

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}$$

Since I_A=0, and from I_B = I_F = equation (69), and I_C = - I_F, we can substitute in and solve for $\mathbf{I_{012}}$:

$$\begin{aligned} I_0 &= 0 \\ I_1 &= \frac{1}{3}\left(a - a^2\right)I_F = \frac{j}{\sqrt{3}}\left(\frac{V_{B,S} - V_{C,S}}{Z_1 + Z_2}\right) \\ I_2 &= \frac{1}{3}\left(a^2 - a\right)I_F = \frac{-j}{\sqrt{3}}\left(\frac{V_{B,S} - V_{C,S}}{Z_1 + Z_2}\right) \end{aligned} \tag{70}$$

Further, (19) tells us that:

$$\begin{aligned} V_{B,S} - V_{C,S} &= \left(a^2 - a\right)V_{1,S} + \left(a - a^2\right)V_{2,S} \\ &= -j\sqrt{3}V_{1,S} + j\sqrt{3}V_{2,S} \end{aligned} \tag{71}$$

Substitution of (71) into (70) gives us I_1 and I_2:

$$\begin{aligned} I_1 &= \frac{V_{1,S} - V_{2,S}}{Z_1 + Z_2} \\ I_2 &= \frac{V_{2,S} - V_{1,S}}{Z_1 + Z_2} = -I_1 \end{aligned} \tag{72}$$

Substituting (71) into (69) gives us the fault current:

$$I_F = \frac{j\sqrt{3}\left(V_{2,S} - V_{1,S}\right)}{Z_1 + Z_2} \tag{73}$$

If $V_{2,S}$ is negligible, this reduces to:

$$\begin{aligned} I_1 &= \frac{V_{1,S}}{Z_1 + Z_2} \\ I_2 &= -I_1 \\ I_F &= \frac{-j\sqrt{3}V_{1,S}}{Z_1 + Z_2} \end{aligned} \tag{74}$$

Again, we can see that the equations derived above give the same answer as the symmetrical component network commonly given for B phase to C phase fault, figure 9.

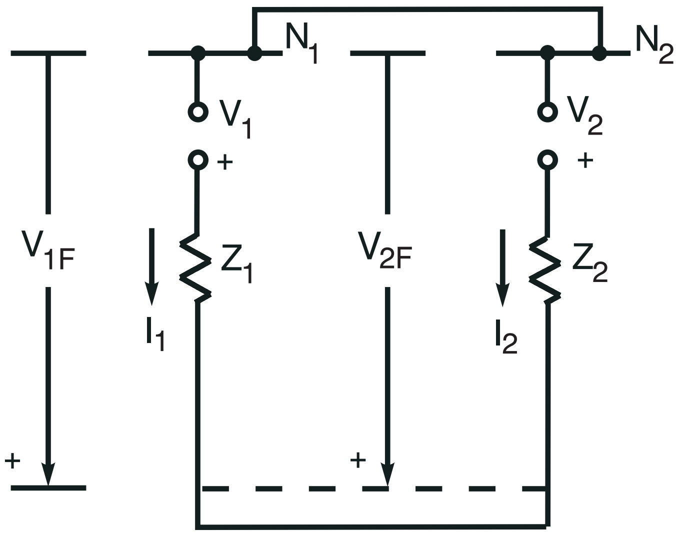

Figure 9 - Symmetrical Component Network - B Phase to C Phase Fault

B to C to Neutral Fault

The ABC domain voltage drop equation for a B to C to neutral fault at the receiving end is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} 0 \\ I_B \\ I_C \end{bmatrix} \tag{75}$$

Faulted System Conditions: The fault currents I_B and I_C are not known and can vary from each other. The current on the unfaulted phase A is 0. The received voltage on phase B and C is zero relative to the local neutral (due to the phase to neutral fault). The received voltage on phase A is unknown. The sending end voltage $\mathbf{V_{ABC,S}}$ is known.

This analysis of this fault and development of the symmetrical component network will be a bit harder to follow. The ABC domain calculations are easy enough to analyze, but the equations are a bit difficult to state in term of sequence component quantities, as will be seen below. Therefore, before solving (75) in the ABC domain, let us first look at developing the symmetrical component network for this problem using the 012 domain voltage drop equation.

Equation (75), when converted to the 012 domain, has the form of:

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{A,R}/3 \\ V_{A,R}/3 \\ V_{A,R}/3 \end{bmatrix} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \tag{76}$$

Note we have applied:

$$\begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} V_{A,R} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_{A,R}/3 \\ V_{A,R}/3 \\ V_{A,R}/3 \end{bmatrix} \tag{77}$$

Restating (76), we see we need a circuit where:

$$\begin{aligned} V_{0,R} &= V_{1,R} = V_{2,R} \\ V_{0,R} &= V_{0,S} - Z_0 I_0 \\ V_{1,R} &= V_{1,S} - Z_1 I_1 \\ V_{2,R} &= V_{2,S} - Z_2 I_2 \end{aligned} \tag{78}$$

Further, at the fault, we know that $I_A = 0$, and from (20) we know that:

$$I_A = I_0 + I_1 + I_2 = 0 \tag{79}$$

Hence, we know that we want to develop a circuit where the sequence currents add together to 0 for all fault conditions.

The symmetrical component network for a B to C to neutral fault is shown in figure 10. Note that the diagram fits the requirements of (78) and (79). An equation for currents when there are three sources in parallel ($V_{0,S}$, $V_{1,S}$, $V_{2,S}$) is a bit involved, but typically, $V_{0,S}$ and $V_{2,S}$ is considered 0, so the equations reduce to a system with a single voltage source and, hence, the equation for currents is easily stated. When $V_{0,S}$ and $V_{2,S}$ is 0, the equations for sequence currents become:

$$\begin{aligned} I_0 &= \frac{Z_2}{Z_0 + Z_2} I_1 \\ I_1 &= \frac{V_{1,S}\left(Z_0 + Z_2\right)}{Z_0 Z_1 + Z_0 Z_2 + Z_1 Z_2} \\ I_2 &= \frac{Z_0}{Z_0 + Z_2} I_1 \end{aligned} \tag{80}$$

We should still examine solving equation (75) in the ABC domain. This will give a more generic solution approach that we might use for A/B/N and A/C/N faults. There are two unknown currents in (75). The effective set of equations for the two unknown currents is:

$$\begin{bmatrix} V_{B,S} \\ V_{C,S} \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} \\ Z_{M-} & Z_S \end{bmatrix}\begin{bmatrix} I_B \\ I_C \end{bmatrix} \tag{81}$$

Referring back to basic matrix theory, using basic Gaussian elimination on a 2x2 matrix of the form:

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{82}$$

gives the following solutions for X_1 and X_2:

$$X_1 = \frac{D_{22}E_1 - D_{12}E_2}{D_{11}D_{22} - D_{12}D_{21}}$$
$$X_2 = \frac{D_{11}E_2 - D_{21}E_1}{D_{11}D_{22} - D_{12}D_{21}} \quad (83)$$

Hence, the equation for I_B and I_C is:

$$I_B = \frac{Z_S V_{B,S} - Z_{M+} V_{C,S}}{Z_S Z_S - Z_{M+} Z_{M-}}$$
$$I_C = \frac{Z_S V_{C,S} - Z_{M-} V_{B,S}}{Z_S Z_S - Z_{M+} Z_{M-}} \quad (84)$$

Once one calculates I_B and I_C and knows $I_A = 0$, then one applies (20) to find I_0, I_1, and I_2. The numbers can be obtained almost instantly via a computer, but hand calculation analysis of (84) and (20) to find the sequence currents would be a slightly tedious process, especially if done more than a couple times. Reference [1] can help with the calculations.

As an exercise in algebraic manipulation some evening when one is having a hard time getting to sleep, work through the process of proving to oneself that equation (84), after 1) substituting in for Z_S, Z_{M+}, and Z_{M-} from (48), then 2) then stating I_B, I_C, $V_{B,S}$, and $V_{C,S}$, in terms of I_{012} and V_{012} from (19) and (20), and then 3) assuming $V_{0,S}$ =0 and $V_{2,S}$ =0, will reduce to (80).

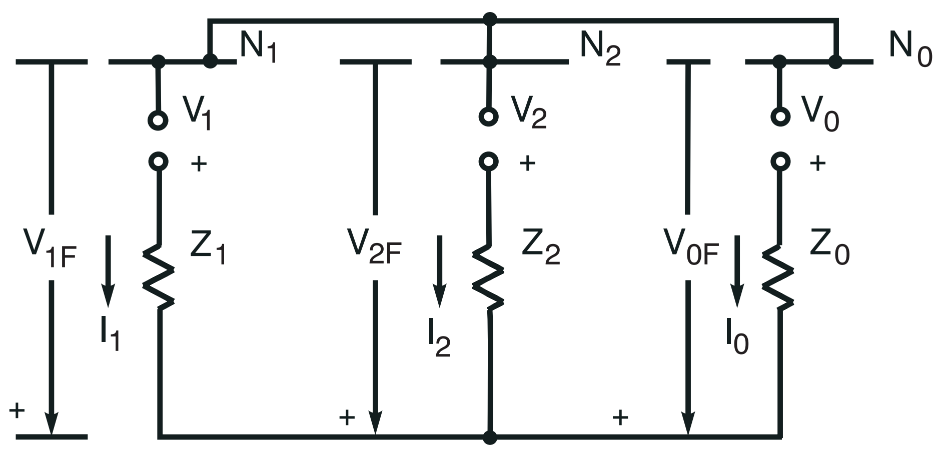

Figure 10 - Symmetrical Component Network - B to C to Neutral Fault

Three Phase Fault

We start by applying (39) to the case of $V_R = 0$ on all phases. The equation becomes:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \quad (85)$$

If the analysis is done in the ABC domain, the equations look like a set of 3 equations with 3 unknowns (I_A, I_B, and I_C), so at first the analysis appears as if it could be complicated. We could proceed to simplify the matter by using our intuition that for this type of fault $I_A = aI_B = a^2I_C$. However, if one works with the 012 voltage drop equation, the analysis is even more straightforward. If we restate (85) in the 012 domain, we obtain the simple-to-analyze equation:

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \quad (86)$$

Or to restate in a more simple form:

$$\begin{aligned} I_0 &= V_{0,S} / Z_0 \\ I_1 &= V_{1,S} / Z_1 \\ I_2 &= V_{2,S} / Z_2 \end{aligned} \quad (87)$$

which agrees with the classical analysis of this fault type, especially for the case where $V_0 = 0$ and $V_2 = 0$. Also, note from (86) that the assumption of balanced impedances is required for this simple answer, since this results in the mutual terms of (86) becoming 0.

General Solution to Shunt Fault Analysis

A general way to calculate the system voltages and currents would be to solve the ABC domain voltage drop equation (16) for the unknown currents. Examine (16) and (26):

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}$$

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix}$$

If we know all the voltages at the sending and receiving ends, the currents can be determined by standard matrix algebra such as (using matrix inverse ($\mathbf{M^{-1}}$) terminology):

$$\begin{aligned} \mathbf{I_{ABC}} &= \left[\mathbf{Z_{ABC}}\right]^{-1}\left[\mathbf{V_{ABC,S}} - \mathbf{V_{ABC,R}}\right] \\ &= \left[\mathbf{Y_{ABC}}\right]\left[\mathbf{V_{ABC,S}} - \mathbf{V_{ABC,R}}\right] \end{aligned}$$

Or: (88)

$$\begin{aligned} \mathbf{I}_{012} &= \left[\mathbf{Z_{012}}\right]^{-1}\left[\mathbf{V}_{012,\mathbf{S}} - \mathbf{V}_{012,\mathbf{R}}\right] \\ &= \left[\mathbf{Y_{012}}\right]\left[\mathbf{V}_{012,\mathbf{S}} - \mathbf{V}_{012,\mathbf{R}}\right] \end{aligned}$$

There is a difficulty in analyzing (88) for fault analysis in that we do not fully know the received voltage, $\mathbf{V_{ABC,R}}$ or $\mathbf{V_{012,R}}$ at the onset of trying to calculate (88). Hence, (88) has to be revised and restated in terms of what is known. In our simple system, in a shunt fault to neutral, we know the received voltages on the faulted phase (0V). In phase to phase faults, we know that the voltage on the two faulted phases is the same. Hence, the general approach to shunt fault analysis varies according to fault type and which phase is faulted. In a shunt fault to neutral, we can reduce (16) to state only those equations that involve the faulted phases and solve for I only on the faulted phase (note the hint that working in the ABC domain may be a bit more intuitive since we intuitively know that if A goes to neutral, $V_{AN} = 0$). Once the fault current is known, voltages on the unfaulted phases can be determined. An example is (52), where phase B and C are eliminated from (16) to solve for I_A, and (81), where phase A is eliminated from (16) and we solve for I_B and I_C. For phase to phase faults, we remove the unfaulted phases from the set of equations (I=0) and then subtract the two equations for the two faulted phases to eliminate the received voltage from the equation, as was done in (68). Once I on the faulted phase(s) is known, we can apply (16) to find voltage on the unfaulted phase and (19) and (20) to state the solutions for current in terms of symmetrical component quantities.

The general approach to fault calculations described above borders on how large scale system-wide models approach the topic of fault calculations. In large scale programs, there are many intermediate buses involved in the fault for which we do not know voltages on any phase, as well as the item of unknown voltages at unfaulted phases at the faulted bus. Further, the effect of delta-wye transformers has to be included. The approach and discussion of how this all works out in a large scale system with many more unknown buses is left to a possible future paper on the topic. Until this paper is written, a paper with enough detail to be possibly understandable to the relatively motivated reader with only a little background in the topic of large scale fault calculations is reference [6].

Series (Open Conductor) Faults

In open conductor faults, the voltages at the two ends of the line are specified and one or more of the conductors is considered open. Upon examination of the resultant voltage drop equation, one finds parallels with the previous shunt fault analysis that will make the open conductor analysis much easier than might be expected.

Since an open conductor tends to imply a transmission line, it would a fair assumption that in most cases that can be envisioned $Z_1 = Z_2$, so the use of Z_{M+} and Z_{M-} may be unnecessary. However, the equations to follow will not be overly complicated by keeping this generality. In fact, toward increasing generality, the equations described below, (89) and (90) with arbitrary $\mathbf{Z_{ABC}}$, could also be utilized for open conductor analysis.

General Solution to Open Circuit and Line Unbalance Faults

Maybe the most straightforward and general way to calculate the effect of an open conductor is by modifying the $\mathbf{Z_{ABC}}$ matrix (14) to reflect a high self impedance (especially R_P) of the open phase and then solving the ABC or 012 domain voltage drop equation (16) or (26) for $\mathbf{I_{ABC}}$ or $\mathbf{I_{012}}$. Since we know the full set of voltages on the two ends of the line, we can work in either the ABC or 012 domain. The equations were noted in (88) above. Expanding them a bit, they take the form of either of the following:

$$\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix}^{-1} \left[\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} \right] \quad (89)$$

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix}^{-1} \left[\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} \right] \quad (90)$$

Solving (89) and (90) is easily done by most computers, but it still can be difficult to get all the equations correct, especially when working with fairly basic tools such as spreadsheets that have user-unfriendly and marginal complex number and matrix calculation capabilities.

Phase A Open

The ABC domain voltage drop equation for a phase A open is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} 0 \\ I_B \\ I_C \end{bmatrix} \quad (91)$$

Faulted System Conditions: The fault current I_B and I_C is unknown and can vary from each other. The current on the open phase A is 0. The sending and receiving voltages are known.

Note the effective equivalence between the equation above and the B to C to neutral shunt fault described by (75). In the B to C to neutral fault, $V_{B,R}$ and $V_{C,R}$ are specified to be 0, so the voltage across the line impedance on phase B and C is simply $V_{B,S}$ and $V_{C,S}$. In the open phase A analysis, the voltage at the receiving end on phase B and C is still defined, but not 0. Hence, the voltage across the line impedances is:

$$\begin{bmatrix} \Delta V_A \\ \Delta V_B \\ \Delta V_C \end{bmatrix} = \begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} \quad (92)$$

Similarly, in the B to C to neutral fault, no current flows in phase A, so there is essentially an open circuit at the receiving end. In the open phase A analysis, the current on phase A is also 0. In theory, we have moved the open circuit from the receiving end to some place in the middle of the line. The actual location of the discontinuity does not affect the resultant current.

One factor that might be different compared to the B to C to neutral fault condition is the significance of the level of assumed V_0 and V_2. In an open conductor condition, it might be a good idea to not neglect these quantities.

Hence, we can analyze the event simply using modified versions of the equations for the phase A to neutral fault. Below are some of the equations from the B to C to neutral analysis. The only difference is that V in the equations has been modified to show that we are dealing with the voltage difference, V_S -V_R. In (93) note that negligible V_2 and V_0 is assumed, as it was in (80).

$$
\begin{aligned}
I_0 &= \frac{Z_2}{Z_1 + Z_2} I_1 \\
I_1 &= \frac{\left(V_{1,S} - V_{1,R}\right)\left(Z_0 + Z_2\right)}{Z_0 Z_1 + Z_0 Z_2 + Z_1 Z_2} \\
I_2 &= \frac{Z_0}{Z_1 + Z_2} I_1
\end{aligned}
\tag{93}
$$

If V_0 and V_2 are not negligible, then study figure 11 to develop the more involved equations that arise when there are 3 voltage sources (V_1,V_0, and V_2) in the network. Equations that solve for I_B and I_C directly from the ABC domain voltage drop equation are:

$$
\begin{aligned}
I_B &= \frac{Z_S\left(V_{B,S} - V_{B,R}\right) - Z_{M+}\left(V_{C,S} - V_{C,R}\right)}{Z_S Z_S - Z_{M+} Z_{M-}} \\
I_C &= \frac{Z_S\left(V_{C,S} - V_{C,R}\right) - Z_{M-}\left(V_{B,S} - V_{B,R}\right)}{Z_S Z_S - Z_{M+} Z_{M-}}
\end{aligned}
\tag{94}
$$

The symmetrical component network to represent the equations is shown in figure 11. The typical practice in developing these diagrams is to divide the line impedance between point X and Y, showing the open condition somewhere in the middle of the line impedances. This approach allows for showing network interconnections and allows using voltage drop analysis to find voltages for faults at intermediate points on the line. In figure 11, Z_1, Z_2, and Z_0 is divided into $Z_{1,X}$ + $Z_{1,Y}$; $Z_{2,X}$ + $Z_{2,Y}$; and $Z_{0,X}$ + $Z_{0,Y}$. Hence, the Z_1 value entered into (93) and (94) is the summation $Z_1 = Z_{X1} + Z_{Y1}$. The same concept applies to Z_2 and Z_0.

In many sources, in open phase networks the voltages are not called V_S and V_R but V_X and V_Y instead. For consistency with the rest of the paper, the figures to follow show the continued use of V_S and V_R.

Also, in these types of drawings, positive current is into the line from both ends, which is counter to the previous assumptions of positive into the line at the sending end and out of the line at the receiving end. This assumption of positive current as into the line at the receiving end has a large impact on how the symmetrical component network is laid out. Hence, note the symmetrical component network in figure 11 is designed to cause $\mathbf{I_{012}}$ current at the sending end and receiving end to be 180^0 out of phase with one another.

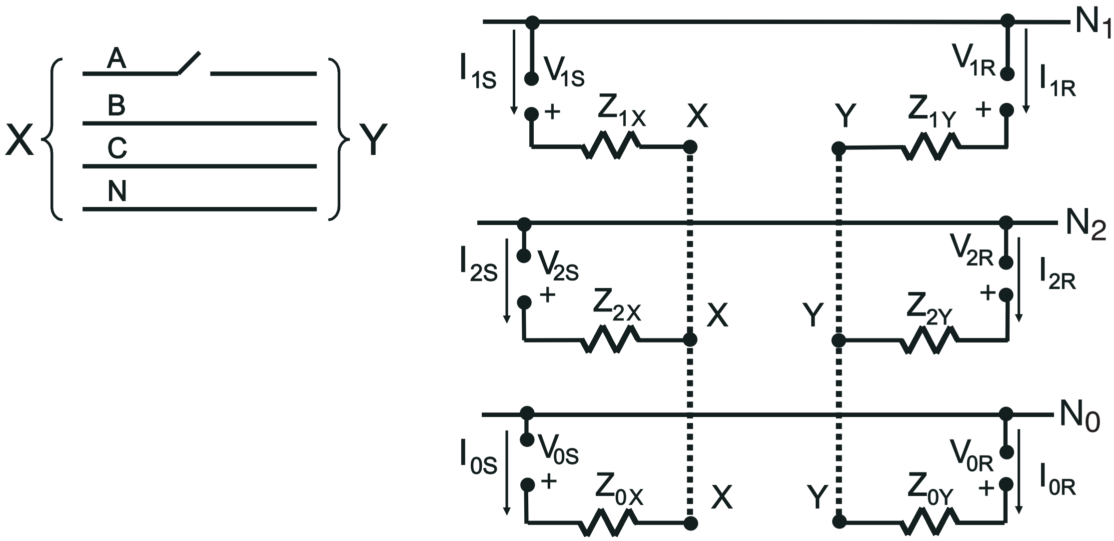

Figure 11 - Symmetrical Component Network - A Phase Open

Phase B and C Open

The ABC domain voltage drop equation for a B and C phase open is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_S & Z_{M+} & Z_{M-} \\ Z_{M-} & Z_S & Z_{M+} \\ Z_{M+} & Z_{M-} & Z_S \end{bmatrix} \begin{bmatrix} I_A \\ 0 \\ 0 \end{bmatrix} \tag{95}$$

Faulted System Conditions: The fault current I_A is unknown. The current on the open phase B and C is 0. The sending and receiving voltages are known.

Note the effective equivalence between the equation above and the A to neutral shunt fault described by (49). In the A to neutral fault, $V_{A,R}$ is specified to be 0, so the voltage across the line impedance on phase A is simply $V_{A,S}$. In the open phase A analysis, the voltage at the receiving end on phase A is still defined, but not 0. Hence, the voltage across the line impedances is as mentioned in (92):

$$\begin{bmatrix} \Delta V_A \\ \Delta V_B \\ \Delta V_C \end{bmatrix} = \begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix}$$

We have simply changed the magnitude of the voltage drop across line A as compared to the shunt fault condition. Also, we are stating no current flows in phase B or C due to the open conductor condition. We have moved the discontinuity in the line from the receiving end to somewhere in the middle of the line.

Hence, we can analyze the event simply using modified versions of the equations for the phase A to neutral fault. Below are some of the equations for the analysis. The only difference is that V in the equations has been modified to show that we are dealing with the voltage difference, V_S - V_R.

$$I_{0,S} = I_{1,S} = I_{2,S} = \frac{\left(V_{0,S} + V_{1,S} + V_{2,S}\right) - \left(V_{0,R} + V_{1,R} + V_{2,R}\right)}{Z_0 + Z_1 + Z_2} \quad (96)$$

In the typical power system, we assume $V_{0,S}$ and $V_{2,S}$ are negligible, so (96) can be reduced to:

$$I_{0,S} = I_{1,S} = I_{2,S} = \frac{V_{1,S} - V_{1,R}}{Z_0 + Z_1 + Z_2} \quad (97)$$

The symmetrical component network to represent the equations is shown in figure 12. Again, the open phase circuit is shown with the line impedance divided between point X and Y, showing the open condition somewhere in the middle of the line impedances. In figure 12, Z_1, Z_2, and Z_0 is divided into $Z_{1,X}$ + $Z_{1,Y}$; $Z_{2,X}$ + $Z_{2,Y}$; and $Z_{0,X}$ + $Z_{0,Y}$. Hence, the Z_1 value entered into (96) and (97) is the summation $Z_1 = Z_{X1} + Z_{Y1}$. The same concept applies to Z_2 and Z_0.

Also, note again positive current is into the line from both ends; therefore, the symmetrical component network is designed to cause current at the sending end and receiving end to be 180^0 out of phase with one another, which results in an odd crisscross of the network connections.

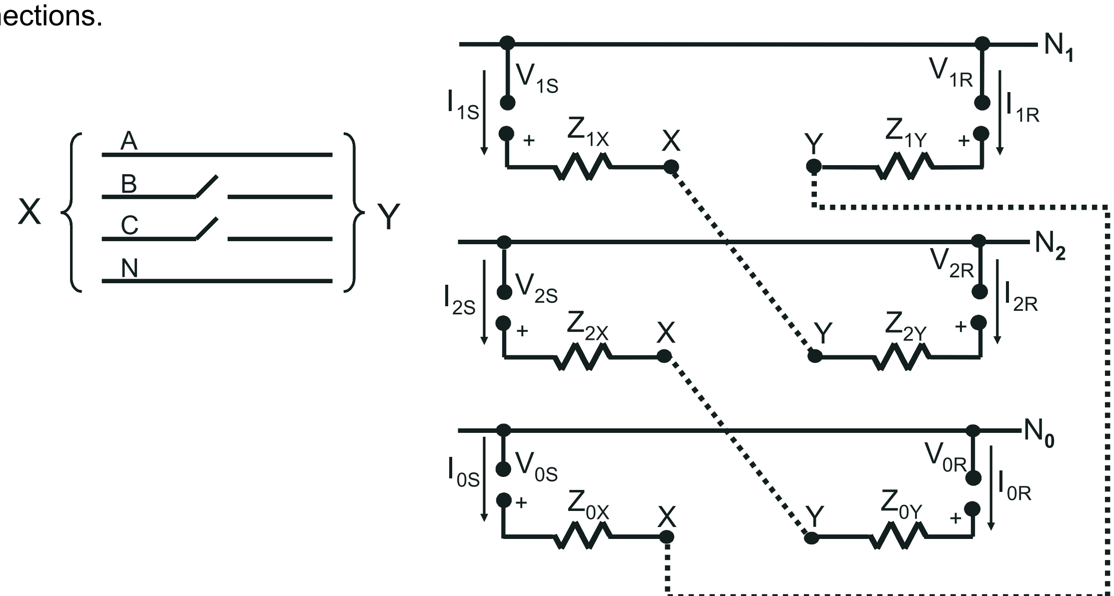

Figure 12 - Symmetrical Component Network - B and C Phase Open

Series Unbalance

This paper will not directly study unbalanced series impedances; it may be included in a future update to the paper. Note a series unbalance will cause the mutual element in the $\mathbf{Z_{012}}$ matrix to become non-zero, as can be seen by reviewing (14) and (30). However, unbalance impedances in the $\mathbf{Z_{ABC}}$ matrix will result in certain forms of the $\mathbf{Z_{012}}$ matrix that some have found modelable via 012 domain circuits. Studying and understanding those 012 domain circuits will be left to the reader. If one wishes to simply analyze the currents that will arise in an unbalanced network for an arbitrary unbalance $\mathbf{Z_{ABC}}$, see (89), (90), and reference [1] for assistance.

Simultaneous Shunt and Series Faults

Due to the high number of permutations of simultaneous faults that might be possible, completely addressing simultaneous shunt and series faults will be left to one example. The equations that have been developed in the paper can be used to address this type of problem, and the reader can extrapolate to determine how to handle other fault types.

Phase A to Neutral with Open to Receiving End

One basic approach is to consider the fault to be the summation of two faults where we have already found a network. Examine figure 13 below. One will see the circuit for an A phase to ground fault via the 1:1 transformer circuit being driven by V_1, V_2, and V_0 at the X node, limited by the X side impedances. If the Y side impedances were infinite, this would reduce to the same network as a phase A to ground fault. If the 1:1 transformer loop were open, then the network would reduce to the network for a phase A open fault. In this figure, we have 7 unknown currents. While there may be ways to reduce and solve this in steps that have a smaller set of currents, the equations to solve for currents in one single set of simultaneous equations might be via something similar to this:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_{1,S} - V_{2,S} + V_{2,R} - V_{1,R} \\ V_{1,S} - V_{0,S} + V_{0,R} - V_{1,R} \\ V_{1,S} + V_{2,S} + V_{0,S} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & Z_{1,X} & -Z_{2,X} & 0 & -Z_{1,Y} & Z_{2,Y} & 0 \\ 0 & Z_{1,X} & 0 & -Z_{0,X} & -Z_{1,Y} & 0 & -Z_{0,Y} \\ 0 & Z_{1,X} & Z_{2,X} & Z_{0,X} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{0,F} \\ I_{1,S} \\ I_{2,S} \\ I_{0,S} \\ I_{1,R} \\ I_{2,R} \\ I_{0,R} \end{bmatrix} \qquad (98)$$

Equation 1: Summation of I at X node
Equation 2: Summation of I at Y node
Equation 3: Summation of I at N_2 node
Equation 4: Summation of I at N_0 node
Equation 5: Voltage drops around loop of $V_{1,X}$, $Z_{1,X}$, $Z_{2,X}$, $Z_{2,Y}$, $Z_{1,Y}$, $V_{1,Y}$
Equation 6: Voltage drops around loop of $V_{1,X}$, $Z_{1,X}$, $Z_{0,X}$, $Z_{0,Y}$, $Z_{1,Y}$, $V_{1,Y}$
Equation 7: Voltage drops around $I_{0,F}$ loop (voltages across 1:1 xfmrs)

This may give us the solution for this particular phase, but how it would be modified for faults on other phases is not intuitive. Let us take an alternate approach to analyzing the fault using the ABC domain. From the sending end to the fault location, the voltage drop equation is:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ V_{B,F} \\ V_{C,F} \end{bmatrix} = \begin{bmatrix} Z_{AA,X} & Z_{AB,X} & Z_{AC,X} \\ Z_{BA,X} & Z_{BB,X} & Z_{BC,X} \\ Z_{CA,X} & Z_{CB,X} & Z_{CC,X} \end{bmatrix} \begin{bmatrix} I_{A,S} \\ I_{B,S} \\ I_{C,S} \end{bmatrix} \qquad (99)$$

Note that $V_{A,F} = 0$. From the receiving end to the fault location, the voltage drop equation is:

$$\begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} - \begin{bmatrix} V_{A,F} \\ V_{B,F} \\ V_{C,F} \end{bmatrix} = \begin{bmatrix} Z_{AA,Y} & Z_{AB,Y} & Z_{AC,Y} \\ Z_{BA,Y} & Z_{BB,Y} & Z_{BC,Y} \\ Z_{CA,Y} & Z_{CB,Y} & Z_{CC,Y} \end{bmatrix} \begin{bmatrix} 0 \\ I_{B,R} \\ I_{C,R} \end{bmatrix} \tag{100}$$

Note that in this case, $V_{A,F}$ is unknown. There is a discontinuity in $V_{A,F}$ due to the voltage across the open circuit. We know $I_{A,R} = 0$, $I_{B,R} = -I_{B,S}$, and $I_{C,R} = -I_{C,S}$. Hence (100) can be reduced to a statement of $I_{B,S}$ and $I_{C,S}$ and solved for $V_{B,F}$ and $V_{C,F}$:

$$\begin{bmatrix} V_{B,F} \\ V_{C,F} \end{bmatrix} = \begin{bmatrix} V_{B,R} \\ V_{C,R} \end{bmatrix} + \begin{bmatrix} Z_{BB,Y} & Z_{BC,Y} \\ Z_{CB,Y} & Z_{CC,Y} \end{bmatrix} \begin{bmatrix} I_{B,S} \\ I_{C,S} \end{bmatrix} \tag{101}$$

Now we substitute (101) into (99) and we have an equation that can be solved for system currents from the sending end side:

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA,X} & Z_{AB,X} & Z_{AC,X} \\ Z_{BA,X} & \left(Z_{BB,X} + Z_{BB,Y}\right) & \left(Z_{BC,X} + Z_{BC,Y}\right) \\ Z_{CA,X} & \left(Z_{CB,X} + Z_{CB,Y}\right) & \left(Z_{CC,X} + Z_{CC,Y}\right) \end{bmatrix} \begin{bmatrix} I_{A,S} \\ I_{B,S} \\ I_{C,S} \end{bmatrix} \tag{102}$$

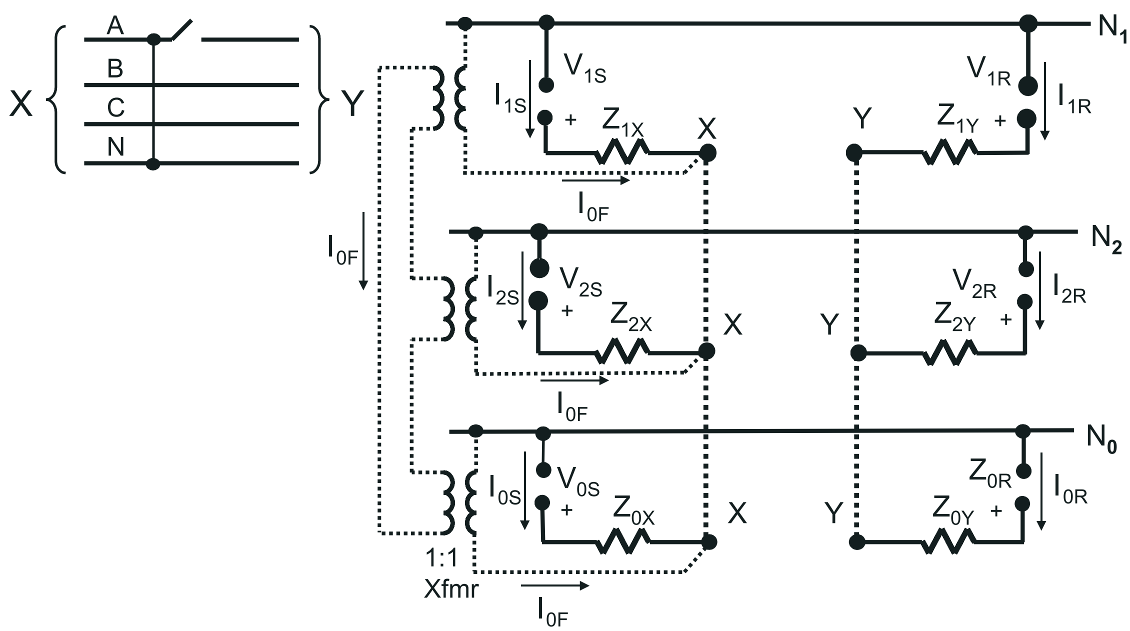

Figure 13 - Symmetrical Component Network - A to Neutral and A Phase Open

The reader should have an idea of how to handle faults on other arbitrary phases from this example. Also from this example, the reader might have enough of the analysis concepts needed to address other simultaneous fault types.

Conclusions

We hope this fills a need in the electric power industry for a paper describing some aspects of the derivation of symmetrical component networks. It should provide a guide for how to handle arbitrary faults that fit into the concept of the basic two port voltage drop equation, $\mathbf{V_S} - \mathbf{V_R} = \mathbf{Z \cdot I}$. It should also give the reader a guide for how one would do basic fault analysis in the ABC domain as well as the 012 domain.

References

1. "ElectricCalcs_R2_0.xls" Spreadsheet to perform various electric system calculations, available at www.basler.com.
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6. Fernando L. Alvarado, Sao Khai Mong, Mark K. Enns, "A Fault Program with Macros, Monitors, and Direct Compensation in Mutual Groups" IEEE Transactions on Power Apparatus and Systems, PAS-104, No. 5, May 1985, p1109-1120

Appendix A - Some Equations for the Basic Faults

This is only a sample of the calculations involved in power system analysis of even the small voltage drop equation analyzed in this paper. For instance, in many relays the phase relationship between sequence voltage and sequence current is compared for directional analysis, a topic not directly covered nor equations provided.

As developed in the paper, the $\mathbf{Z_{ABC}}$ elements below are the appropriate Z_S, Z_{M+}, and Z_{M-} quantities, identified in (44) and (48).

1) Phase A to Neutral

Note that the equations below also are suitable for analyzing an B and C phase open condition, where $V_{B,R}$ is not 0. See discussion in paper in section on open phase analysis.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_F \\ 0 \\ 0 \end{bmatrix} \Rightarrow V_{A,S} = Z_{AA} I_F$$

$$I_F = I_A = \frac{V_{A,S}}{Z_S} = \frac{3V_{A,S}}{Z_0 + Z_1 + Z_2} = \frac{3\left(V_{0,S} + V_{1,S} + V_{2,S}\right)}{Z_0 + Z_1 + Z_2}$$

$$I_0 = I_1 = I_2 = \frac{V_{0,S} + V_{1,S} + V_{2,S}}{Z_0 + Z_1 + Z_2}$$

2) Phase B to Neutral

Note that the equations below also are suitable for analyzing an A and C phase open condition, where $V_{B,R}$ is not 0. See discussion in paper in section on open phase analysis.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ 0 \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} 0 \\ I_F \\ 0 \end{bmatrix} \Rightarrow V_{B,S} = Z_{BB} I_F$$

$$I_F = I_B = \frac{V_{B,S}}{Z_S} = \frac{3 \cdot V_{B,S}}{Z_0 + Z_1 + Z_2} = \frac{3\left(V_{0,S} + a^2 V_{1,S} + aV_{2,S}\right)}{Z_0 + Z_1 + Z_2}$$

$$I_0 = \frac{V_{0,S} + a^2 V_{1,S} + aV_{2,S}}{Z_0 + Z_1 + Z_2} = a^2 I_1$$

$$I_1 = \frac{aV_{0,S} + V_{1,S} + a^2 V_{2,S}}{Z_0 + Z_1 + Z_2}$$

$$I_2 = \frac{a^2 V_{0,S} + aV_{1,S} + V_{2,S}}{Z_0 + Z_1 + Z_2} = aI_1$$

3) Phase C to Neutral

Note that the equations below also are suitable for analyzing an A and B phase open condition, where $V_{C,R}$ is not 0. See discussion in paper in section on open phase analysis.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I_F \end{bmatrix} \Rightarrow V_{C,S} = Z_{CC} I_F$$

$$I_F = I_C = \frac{V_{C,S}}{Z_S} = \frac{3 \cdot V_{C,S}}{Z_0 + Z_1 + Z_2} = \frac{3\left(V_{0,S} + aV_{1,S} + a^2 V_{2,S}\right)}{Z_0 + Z_1 + Z_2}$$

$$I_0 = \frac{V_{0,S} + aV_{1,S} + a^2 V_{2,S}}{Z_0 + Z_1 + Z_2} = aI_1$$

$$I_1 = \frac{a^2 V_{0,S} + V_{1,S} + aV_{2,S}}{Z_0 + Z_1 + Z_2}$$

$$I_2 = \frac{aV_{0,s} + a^2 V_{1,S} + V_{2,S}}{Z_0 + Z_1 + Z_2} = a^2 I_1$$

4) Phase A to Phase B

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_F \\ V_F \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_F \\ -I_F \\ 0 \end{bmatrix} \Rightarrow V_{A,S} - V_{B,S} = \left(Z_{AA} + Z_{BB} - Z_{AB} - Z_{BA}\right) I_F$$

$$I_F = \frac{\left(1-a^2\right)V_{1,S} + \left(1-a\right)V_{2,S}}{Z_1 + Z_2}$$

$$I_1 = \frac{V_{1,S} - aV_{2,S}}{Z_1 + Z_2}$$

$$I_2 = \frac{V_{2,S} - a^2 V_{1,S}}{Z_1 + Z_2}$$

5) Phase B to Phase C

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_F \\ V_F \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} 0 \\ I_F \\ -I_F \end{bmatrix} \Rightarrow V_{B,S} - V_{C,S} = \left(Z_{BB} + Z_{CC} - Z_{BC} - Z_{CB}\right) I_F$$

$$I_F = \frac{j\sqrt{3}\left(V_{2,S} - V_{1,S}\right)}{Z_1 + Z_2}$$

$$I_1 = \frac{V_{1,S} - V_{2,S}}{Z_1 + Z_2}$$

$$I_2 = \frac{V_{2,S} - V_{1,S}}{Z_1 + Z_2} = -I_1$$

6) Phase C to Phase A

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_F \\ V_{B,R} \\ V_F \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} -I_F \\ 0 \\ I_F \end{bmatrix} \Rightarrow V_{C,S} - V_{A,S} = \left(Z_{AA} + Z_{CC} - Z_{AC} - Z_{CA}\right) I_F$$

$$I_F = \frac{\left(a-1\right)V_{1,S} + \left(a^2-1\right)V_{2,S}}{Z_1 + Z_2}$$

$$I_1 = \frac{V_{1,S} - a^2 V_{2,S}}{Z_1 + Z_2}$$

$$I_2 = \frac{V_{2,S} - aV_{1,S}}{Z_1 + Z_2}$$

7) Phase A to B to Neutral

Note that the equations below also are suitable for analyzing a phase C open condition, where $V_{A,R}$ and $V_{C,R}$ are not 0. See discussion in paper on open phase conditions.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} V_{A,S} \\ V_{B,S} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} \\ Z_{BA} & Z_{BB} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix}$$

$$I_A = \frac{Z_{BB} V_{A,S} - Z_{AB} V_{B,S}}{Z_{AA} Z_{BB} - Z_{AB} Z_{BA}}$$

$$I_B = \frac{Z_{AA} V_{B,S} - Z_{BA} V_{A,S}}{Z_{AA} Z_{BB} - Z_{AB} Z_{BA}}$$

8) Phase B to C to Neutral

Note that the equations below also are suitable for analyzing a phase A open condition, where $V_{B,R}$ and $V_{C,R}$ are not 0.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} 0 \\ I_B \\ I_C \end{bmatrix} \Rightarrow \begin{bmatrix} V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{BB} & Z_{BC} \\ Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_B \\ I_C \end{bmatrix}$$

$$I_A = \frac{Z_{CC} V_{B,S} - Z_{BC} V_{C,S}}{Z_{BB} Z_{CC} - Z_{BC} Z_{CB}}$$

$$I_B = \frac{Z_{BB} V_{C,S} - Z_{CB} V_{B,S}}{Z_{BB} Z_{CC} - Z_{BC} Z_{CB}}$$

Also, if $V_{2,S} = 0$ and $V_{0,S} = 0$, the following was developed in the paper:

$$I_0 = \frac{Z_2}{Z_0 + Z_2} I_1$$

$$I_1 = \frac{V_{1,S}\left(Z_0 + Z_2\right)}{Z_0 Z_1 + Z_0 Z_2 + Z_1 Z_2}$$

$$I_2 = \frac{Z_0}{Z_0 + Z_2} I_1$$

9) Phase C to A to Neutral

Note that the equations below also are suitable for analyzing a phase B open condition, where $V_{A,R}$ and $V_{C,R}$ are not 0.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ V_{B,R} \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ 0 \\ I_C \end{bmatrix} \Rightarrow \begin{bmatrix} V_{A,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AC} \\ Z_{CA} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_C \end{bmatrix}$$

$$I_A = \frac{Z_{CC} V_{A,S} - Z_{AC} V_{C,S}}{Z_{AA} Z_{CC} - Z_{AC} Z_{CA}}$$

$$I_C = \frac{Z_{AA} V_{C,S} - Z_{CA} V_{A,S}}{Z_{AA} Z_{CC} - Z_{AC} Z_{CA}}$$