Zero Sequence Impedance of Overhead Transmission Lines

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The positive and negative sequence impedance of overhead transmission lines is a fairly well documented matter in a number of commonly used textbooks on power system analysis. The zero sequence impedance calculations are less widely covered. This paper will review the transmission line impedances, centering on zero sequence impedances, and review the work of John Carson on the topic of ground current impedances.

Summary and Outline

1. First, we review the basic 3x3 $\mathbf{Z_{ABC}}$ domain impedance matrix used to represent the impedance to current flow on a basic 3 phase line, including allowance for earth return.
2. We show how the $\mathbf{Z_{ABC}}$ matrix expands when overhead ground wires or other nearby metallic ground current paths exist (referred to herein as the $\mathbf{Z_{ABC+}}$ matrix). We show how the matrix, once constructed, is reduced in size to the 3x3 $\mathbf{Z_{ABC}}$ matrix.
3. Once one has a 3x3 $\mathbf{Z_{ABC}}$ matrix, we will show how this matrix is converted to a sequence domain $\mathbf{Z_{012}}$ matrix, in which we find Z_{00}, Z_{11}, and Z_{22}.
4. Next, we will show how one calculates the elements of the $\mathbf{Z_{ABC}}$ or $\mathbf{Z_{ABC+}}$ matrices. There are 3 parts:
   a. First, we show how one builds the $\mathbf{Z_{ABC+}}$ matrix from impedances we will find in parts b and c.
   b. We show the calculation of the impedances for cases where current flows strictly in metallic conductors.
   c. We show the calculation of the impedances for the case where current flows in the ground. This impedance is calculated using what is referred to as Carson's equations. This is a fairly complicated section. John Carson's paper has minimal documentation of his thought processes, and to follow his thoughts one has to have an understanding of electromagnetic wave theory. One will find, as the review progresses, that there are un-answered questions, inviting feedback from the readers.

Special Acknowledgements:

It is apparent that John Carson and his 1926 paper are acknowledged [Ref 1]. However, the second acknowledgment, [Ref. 2] is to Michael James O'Conner and his 1987 Master's thesis where he applies Carson's equations to underground cable. I struggled to understand some elements of Carson's equations until the light was shed on the topic by Mr. O'Conner's thesis. Also, I need to acknowledge Mr. O'Conner's thesis advisor, Dr. Satish Ranade (Klipsch School of Electrical and Computer Engineering, New Mexico State University, Las Cruces NM) who, of course, had much input on the thesis and made a copy available to me. Special thanks also to Basler Electric for allowing me the time to prepare this.

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1. Basic Impedance Concepts

In Figure 1 there are three phase current loops, A, B, and C, each passing through a common neutral/ground. The phase A loop is shown via a dotted line. Each phase loop in Figure 2 has a different impedance, because each is defined by a different current path, a different loop cross section area and, when magnetic core material is involved, a different permeability of the material through which the flux passes. To keep the drawing from becoming exceedingly complex, only the main representative flux loops are shown and only for phase A. One's imagination should be used to fill in the blanks.

Current in each loop is assumed to return through the neutral/ground conductor. Since I_G is a summation of I_A, I_B, and I_C, the voltage drop equations eventually will be reduced to remove I_G. Note the assumed direction for positive neutral current. In some text and articles, the reference direction for I_G is reversed which, in turn, can cause negative signs appear in some of the voltage drops equations or impedances.

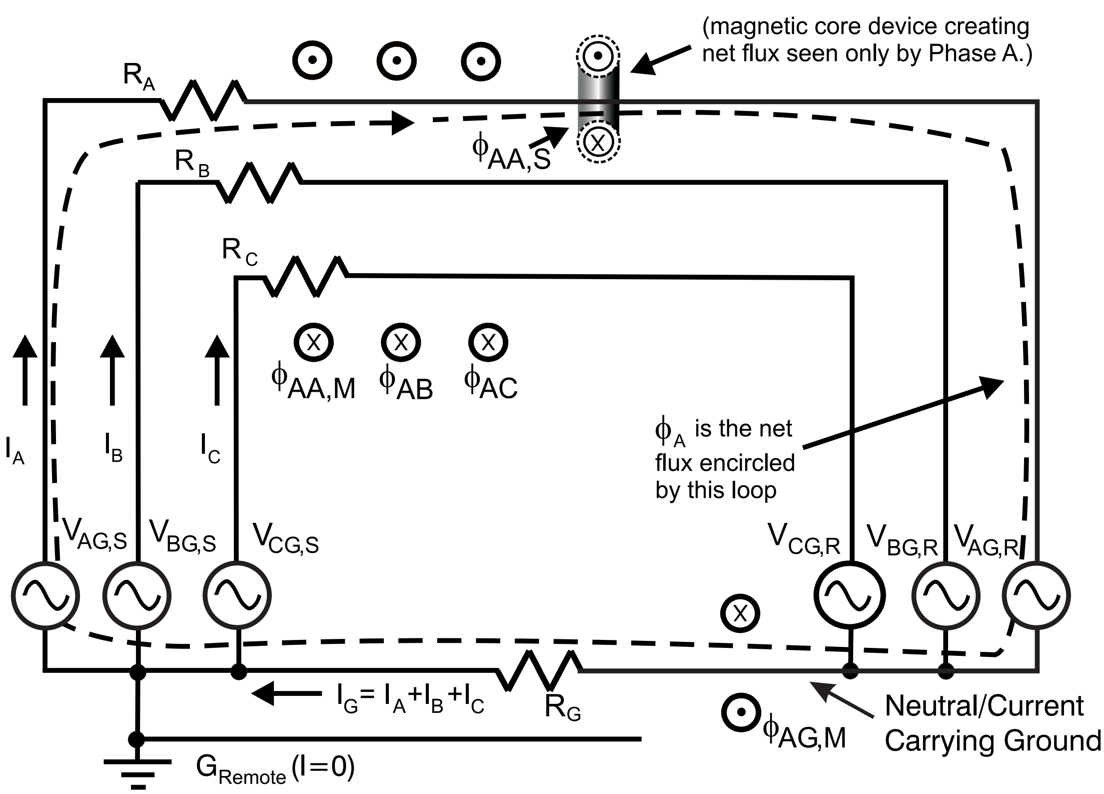

Figure 1 - Phase A Voltage Loop and Flux when part of a Three Phase System

To understand the system impedances that will represent the system in Figure 1, one first should understand the sources of flux that can be found in loop A in Figure 1. The flux in loop A is the summation of the flux generated by the 4 currents specified in Figure 1:

$$\varphi_A = \varphi_{AA} + \varphi_{AB} + \varphi_{AC} + \varphi_{AG} \qquad (1)$$

$$\phi_{AA} = \phi_{AA,S} + \phi_{AA,M}$$

$\phi_{AA,S}$ = Flux due to I_A that is only seen by loop A. Consider this to be a "self" flux. Large self flux can be generated by magnetic core devices. The magnetic core device shown in the diagram has flux that is seen only by loop A and not by loop B or C.

$\phi_{AA,M}$ = Flux in loop A that is due to I_A but that is also seen by loop B and/or C. This flux is the source of mutual inductance of phase A to phases B and C. The mutual flux to each loop B and C would, in practice, be slightly different.

ϕ_{AB} = Flux in loop A due to current I_B. This is the source of mutual coupling between I_B and phase A.

ϕ_{AC} = Flux in loop A due to current I_C. This is the source of mutual coupling between I_C and phase A.

ϕ_{AG} = Flux in loop A due to current I_G. This is the source of mutual coupling between I_G and phase A.

This pattern will exist for the loop associated with each phase.

In an even more complete drawing, not shown in Figure 1, the current in I_G will flow in a slightly differently located effective ground conductor depending on whether the source for I_G is I_A, I_B, or I_C, which will result in varying Φ_{AG} for each source of I_G and introducing a small amount of $\phi_{GG,S}$, and there may be a separate ground conductor that carries current rather than the earth/ground. However, let us try to accept for now the assumption of a single common ground return path as "close enough" for our initial investigation. If one studies Figure 1, the only way to have flux generated by I_G that does not pass through at least one of the A, B, or C phase loops is for the flux to be completely outside of the A, B, and C loops. If flux is outside the A, B, and C loops, then the flux that is generated will not be seen by the voltage sources.

The flux distribution indicated by Figure 1 can be restated in terms of L, and ground current can be removed from the equations:

$$\begin{aligned}\varphi_A &= L_{AA} i_A(t) + L_{AB} i_B(t) + L_{AC} i_C(t) + L_{AG} i_G(t) \\ &= L_{AA} i_A(t) + L_{AB} i_B(t) + L_{AC} i_C(t) + L_{AG}\left(i_A(t) + i_B(t) + i_C(t)\right) \ . \\ &= \left(L_{AA} + L_{AG}\right) i_A(t) + \left(L_{AB} + L_{AG}\right) i_B(t) + \left(L_{AC} + L_{AG}\right) i_C(t)\end{aligned} \tag{2}$$

This flux distribution, when coupled with resistive voltage drop, becomes the basis for system impedances.

There are two basic forms of impedance: "self" impedance and "mutual" impedance. Self impedance refers to the voltage drop in a phase loop due to current in that same phase. For instance, Z_{AA} relates the voltage loss in loop A due to I_A. The instantaneous voltage equation for current only in phase A is

$$v_{AG,S}(t) - v_{AG,R}(t) = \left(R_A + R_G\right) i_A(t) + \left(L_{AA} + L_{AG}\right)\frac{d}{dt} i_A(t) \ . \tag{3}$$

Restating (3) in phasor analysis terms gives us

$$V_{AG,S} - V_{AG,R} = \left(R_A + R_G\right) I_A + j\left(X_{AA} + X_{AG}\right) I_A \ . \tag{4}$$

The equation for self impedance becomes

$$Z_{AA} = \frac{V_{AG,S} - V_{AG,R}}{I_A} \tag{5}$$
$$= (R_A + R_G) + j(X_{AA} + X_{AG}) \quad .$$

Mutual impedance refers to the voltage drop in a loop due to current in another phase. For instance, Z_{AB} relates the voltage loss in loop A due to I_B which, in phasor analysis terms, becomes restated to

$$V_{AG,S} - V_{AG,R} = R_G I_B + j(X_{AB} + X_{AG}) I_B \quad . \tag{6}$$

So the equation for mutual impedance becomes

$$Z_{AB} = \frac{V_{AG,S} - V_{AG,R}}{I_B} \tag{7}$$
$$= (R_G) + j(X_{AB} + X_{AG}) \quad .$$

Now we can define the voltage drop equations that describe the circuit in Figure 1. However, we have one other modification to make to our equation. As previously stated, the ground current loop takes slightly different paths depending on where the overhead conductor is. The figure does not express this matter well. We cannot account for this without a substantial increase in complexity in network, since the three ground current paths tend to overlay one-another. However, we will make some allowance, as seen below, offering a possibility of entering different R_{AG}, R_{BG}. and R_{CG}.

$$\begin{bmatrix} V_{AG,S} \\ V_{BG,S} \\ V_{CG,S} \\ 0 \end{bmatrix} - \begin{bmatrix} V_{AG,R} \\ V_{BG,R} \\ V_{CG,R} \\ 0 \end{bmatrix} = \begin{bmatrix} R_A + X_{AA} & X_{AB} & X_{AC} & R_{AG} + X_{AG} \\ X_{BA} & R_B + X_{BB} & X_{BC} & R_{BG} + X_{BG} \\ X_{CA} & X_{CB} & R_C + X_{CC} & R_{CG} + X_{CG} \\ * & * & * & * \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \\ I_G \end{bmatrix} \tag{8}$$

The * above indicates that these elements could not be well defined in this matrix for the system shown in figure 1. The path modeled by R_G is not contained in an independent ground current loop; it is already part of the A, B, and C loops. Also note that without a ground loop independent from the A, B, and C loops, then the concept of ground loop self impedance, X_{GG} does not fit into our model. To address these problems, we will eliminate I_G from the equations; note that $I_G = I_A + I_B + I_C$, which means we can rewrite our equations as:

$$\begin{bmatrix} V_{AG,S} \\ V_{BG,S} \\ V_{CG,S} \\ 0 \end{bmatrix} - \begin{bmatrix} V_{AG,R} \\ V_{BG,R} \\ V_{CG,R} \\ 0 \end{bmatrix} = \begin{bmatrix} R_A + X_{AA} & X_{AB} & X_{AC} & R_{AG} + X_{AG} \\ X_{BA} & R_B + X_{BB} & X_{BC} & R_{BG} + X_{BG} \\ X_{CA} & X_{CB} & R_C + X_{CC} & R_{CG} + X_{CG} \\ * & * & * & * \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \\ I_A + I_B + I_C \end{bmatrix} \tag{9}$$

which reduces to a set of equations that represents Figure 1,

$$\begin{bmatrix} V_{AG,S} \\ V_{BG,S} \\ V_{CG,S} \end{bmatrix} - \begin{bmatrix} V_{AG,R} \\ V_{BG,R} \\ V_{CG,R} \end{bmatrix} =$$

$$\begin{bmatrix} R_A + R_{AG} + j(X_{AA} + X_{AG}) & R_{AG} + j(X_{AB} + X_{AG}) & R_{AG} + j(X_{AC} + X_{AG}) \\ R_{BG} + j(X_{BA} + X_{BG}) & R_B + R_{BG} + j(X_{BB} + X_{BG}) & R_{BG} + j(X_{BC} + X_{BG}) \\ R_{CG} + j(X_{CA} + X_{CG}) & R_{CG} + j(X_{CB} + X_{CG}) & R_C + R_{BG} + j(X_{CC} + X_{CG}) \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}. \tag{10}$$

A shorthand way of stating (10) is

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}. \tag{11}$$

Equation (11) can be abbreviated further to

$$\mathbf{V_{ABC,S}} - \mathbf{V_{ABC,R}} = \mathbf{Z_{ABC}} \cdot \mathbf{I_{ABC}}. \tag{12}$$

Note in the above that:

- Wye voltages that reference neutral are implied.
- Positive current is implied to be from the sending end to the receiving end.
- The voltage to remote ground is neither utilized nor calculated explicitly.
- In the abbreviated equation (16) the “G” reference for voltages is assumed and not explicitly stated. The G reference will be dropped in the rest of the paper.
- Note the implied definition of $\mathbf{Z_{ABC}}$:

$$\mathbf{Z_{ABC}} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \tag{13}$$

where:

$Z_{AA} = V_{AS}\text{-}V_{AR} / I_A$	$Z_{AB} = V_{AS}\text{-}V_{AR} / I_B$	$Z_{AC} = V_{AS}\text{-}V_{AR} / I_C$
$Z_{BA} = V_{BS}\text{-}V_{BR} / I_A$	$Z_{BB} = V_{BS}\text{-}V_{BR} / I_B$	$Z_{BC} = V_{BS}\text{-}V_{BR} / I_C$
$Z_{CA} = V_{CS}\text{-}V_{CR} / I_A$	$Z_{CB} = V_{CS}\text{-}V_{CR} / I_B$	$Z_{CC} = V_{CS}\text{-}V_{CR} / I_C$

Restating the matter of self and mutual impedances, the terms Z_{AA}, Z_{BB}, and Z_{CC} are referred to as self impedances (e.g., Z_{AA} is a measure that relates phase A voltage and phase A current), and all other inter-phase quantities are referred to as mutual impedances (e.g., Z_{AB} is a measure that interrelates phase A voltage and phase B current).

Figure 2 is a lumped parameter approach to viewing all the impedances described in (13), as well as elsewhere above.

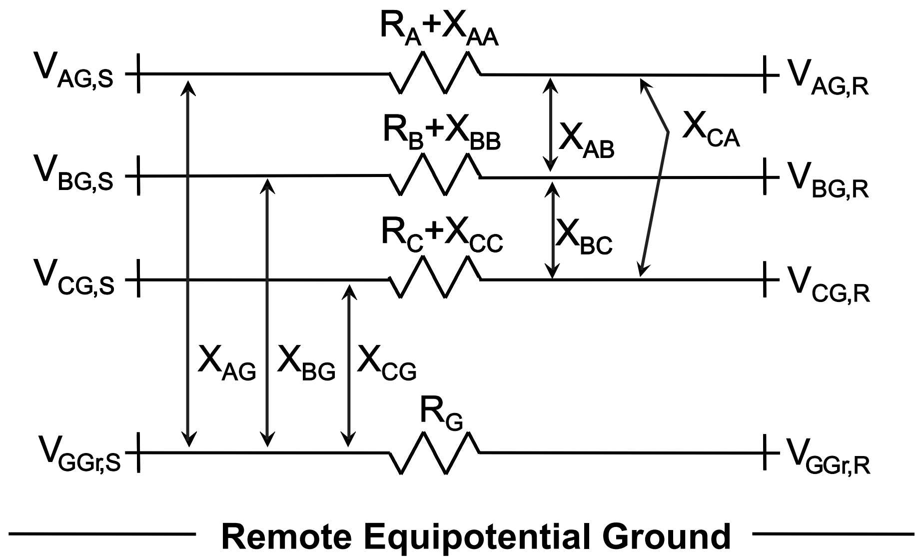

Figure 2 - Lumped Parameter 3 Phase Impedance Network

As a side note, while the only convenient way to show the inductance of an electric circuit is via a lumped parameter on a conductor, this method can be technically a little bit misleading, possibly giving erroneous perceptions of the nature of inductive impedances. The lumped inductance diagram leaves the impression that a wire, all by itself, has inductance. Actually, inductance is not defined until the conductive route back to the source is defined which, in turn, defines the area of the loop and the flux inside the loop.

2. Impedance Matrix for the Case of Metallic Ground Conductors

The above equations were based upon a single ground path. In many, if not most, power lines, there is a ground wire hung along the same right of way as the phase conductors. Sometimes, there may be even two ground wires and a railroad track or similar metallic conductor that can act as a ground conductor. Each parallel path for the ground current has a very different impedance relative to the earth/soil. This case creates an expansion of the impedance matrix. Assume we have two ground paths, a neutral overhead wire, and the ground. For this case, to model each ground current path, we need an impedance matrix with a format very similar to (9), but where we add one more line to the matrix for the voltage loop associated with our ground wire.

$$\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_A + X_{AA} & X_{AB} & X_{AC} & X_{AN} & R_{AG} + X_{AG} \\ X_{BA} & R_B + X_{BB} & X_{BC} & X_{BN} & R_{BG} + X_{BG} \\ X_{CA} & X_{CB} & R_C + X_{CC} & X_{CN} & R_{CG} + X_{CG} \\ X_{NA} & X_{NB} & X_{NC} & R_N + X_{NN} & R_{NG} + X_{NG} \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \\ I_N \\ I_A + I_B + I_C + I_N \end{bmatrix}. \quad (14)$$

We are not including the “j” term to simplify the matrix a little. Note that the applied voltage to the ground conductor is 0, since it is grounded at both ends. Next, note $I_G = I_A + I_B + I_C + I_N$. This matrix can be reduced to a 4x4 matrix.

$$
\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \\ 0 \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \\ 0 \end{bmatrix} = \tag{15}
$$

$$
\begin{bmatrix} R_A + R_{AG} + X_{AA} + X_{AG} & R_{AG} + X_{AB} + X_{AG} & R_{AG} + X_{AC} + X_{AG} & R_{AG} + X_{AN} + X_{AG} \\ R_{BG} + X_{BA} + X_{BG} & R_B + R_{BG} + X_{BB} + X_{BG} & R_{BG} + X_{BC} + X_{BG} & R_{BG} + X_{BN} + X_{BG} \\ R_{CG} + X_{CA} + X_{CG} & R_{CG} + X_{CB} + X_{CG} & R_C + R_{CG} + X_{CC} + X_{CG} & R_{CG} + X_{CN} + X_{CG} \\ R_{NG} + X_{NA} + X_{NG} & R_{NG} + X_{NB} + X_{NG} & R_{NG} + X_{NC} + X_{NG} & R_N + R_{NG} + X_{NN} + X_{NG} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \\ I_N \end{bmatrix} .
$$

The benefit of this equation is that it calculates I_N directly, but the problem is that it does not have the format of the standard 3x3 impedance matrix we need for classical circuit analysis and calculation of Z_0, Z_1, and Z_2. We need to reduce this to the standard 3x3 matrix by eliminating I_N from the analysis. The process to remove I_N from the equations is a matrix manipulation process called Kron Reduction. Reference [3] and [4] provide the calculation process. The resulting matrix has a large number of terms, so we will show the calculations symbolically.

Given the 4x4 matrix:

$$
\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \\ 0 \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} & Z_{AN} \\ Z_{BA} & Z_{BB} & Z_{BC} & Z_{BN} \\ Z_{CA} & Z_{CB} & Z_{CC} & Z_{CN} \\ Z_{NA} & Z_{NB} & Z_{NC} & Z_{NN} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \\ I_N \end{bmatrix} \tag{16}
$$

then

$$
\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} - \frac{Z_{AN} Z_{NA}}{Z_{NN}} & Z_{AB} - \frac{Z_{AN} Z_{NB}}{Z_{NN}} & Z_{AC} - \frac{Z_{AN} Z_{NC}}{Z_{NN}} \\ Z_{BA} - \frac{Z_{BN} Z_{NA}}{Z_{NN}} & Z_{BB} - \frac{Z_{BN} Z_{NB}}{Z_{NN}} & Z_{BC} - \frac{Z_{BN} Z_{NC}}{Z_{NN}} \\ Z_{CA} - \frac{Z_{CN} Z_{NA}}{Z_{NN}} & Z_{CB} - \frac{Z_{CN} Z_{NB}}{Z_{NN}} & Z_{CC} - \frac{Z_{CN} Z_{NC}}{Z_{NN}} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} . \tag{17}
$$

If one had more than one additional ground path, one would repeat this reduction the appropriate number of times.

3. Representing Z_{ABC} as Z_{012} Sequence Impedances

A fairly concise review of symmetrical component theory is available in [6], but [3] and [4] are more complete resources.

The equations above were for real world impedances. To convert them to the positive, negative, and zero sequence impedances, recall the ABC domain voltage drop equation

$$
\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} . \tag{18}
$$

Recall also the matrices that are used in the conversion from ABC to 012 domain, and back.

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \tag{19}$$

One should recall that in matrix mathematics, similar to algebraic manipulation, if we multiply all elements on both sides of the equation by the same matrix, we have not have changed the equality of the two sides of the equation. In this case we will multiply across by $\mathbf{A}^{-1}$. Further, if we multiply by $\mathbf{A}\cdot\mathbf{A}^{-1}$ we have effectively only multiplied by 1, so a valid modification of (18) is

$$\mathbf{A}^{-1}\begin{bmatrix} V_{A,S} \\ V_{B,S} \\ V_{C,S} \end{bmatrix} - \mathbf{A}^{-1}\begin{bmatrix} V_{A,R} \\ V_{B,R} \\ V_{C,R} \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix}\mathbf{A}\cdot\mathbf{A}^{-1}\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}. \tag{20}$$

This equation, when multiplied out, will become the 012 domain voltage drop equation. The reader should already recognize the conversion of $\mathbf{V_{ABC}}$ and $\mathbf{I_{ABC}}$ to $\mathbf{V_{012}}$ and $\mathbf{I_{012}}$ from previous experience. Note, however, that we are converting the $\mathbf{Z_{ABC}}$ domain matrix to the $\mathbf{Z_{012}}$ domain as well. The form of the resultant equation will be

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix}. \tag{21}$$

which can be abbreviated further to

$$\mathbf{V}_{012,\mathbf{S}} - \mathbf{V}_{012,\mathbf{R}} = \mathbf{Z}_{012}\cdot\mathbf{I}_{012}\ . \tag{22}$$

Note the definition of the $\mathbf{Z_{012}}$ impedance matrix elements is

$$\mathbf{Z}_{012} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix}. \tag{23}$$

where:

$Z_{00} = (V_{0,S} - V_{0,R}) / I_0$ $\quad Z_{01} = (V_{0,S} - V_{0,R}) / I_1$ $\quad Z_{02} = (V_{0,S} - V_{0,R}) / I_2$

$Z_{10} = (V_{1,S} - V_{1,R}) / I_0$ $\quad Z_{11} = (V_{1,S} - V_{1,R}) / I_1$ $\quad Z_{12} = (V_{1,S} - V_{1,R}) / I_2$

$Z_{20} = (V_{2,S} - V_{2,R}) / I_0$ $\quad Z_{21} = (V_{2,S} - V_{2,R}) / I_1$ $\quad Z_{22} = (V_{2,S} - V_{2,R}) / I_2$

The equation to convert from $\mathbf{Z_{ABC}}$ to $\mathbf{Z_{012}}$ is the $\mathbf{A^{-1}Z_{ABC}A}$ portion of (20) is

$$\mathbf{Z}_{012} = \mathbf{A}^{-1}\begin{bmatrix} Z_{AA} & Z_{AB} & Z_{AC} \\ Z_{BA} & Z_{BB} & Z_{BC} \\ Z_{CA} & Z_{CB} & Z_{CC} \end{bmatrix}\mathbf{A} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix}. \tag{24}$$

The Zero Sequence impedance of the line will be Z_{00} in the matrix above.
The text above is highlighted just to make it clear that we have finally reached the topic and title of this paper. Similarly, the positive sequence impedance will be Z_{11} and negative sequence impedance will be Z_{22}. This process of calculating $\mathbf{Z_{012}}$ from the individual elements, rather than using the more commonly reported approach of a GMD of the set of phase conductors, will give us the resulting non-ideal nature of the impedances; e.g., any off diagonal element in (23) will cause cross coupling of the symmetrical component networks.

The set of equations implied by the matrix math of (24) is

$$\begin{aligned}
Z_{00} &= \left(Z_{AA}+Z_{BA}+Z_{CA}+Z_{AB}+Z_{BB}+Z_{CB}+Z_{AC}+Z_{BC}+Z_{CC}\right)/3\\
Z_{01} &= \left(Z_{AA}+Z_{BA}+Z_{CA}+a^2Z_{AB}+a^2Z_{BB}+a^2Z_{CB}+aZ_{AC}+aZ_{BC}+aZ_{CC}\right)/3\\
Z_{02} &= \left(Z_{AA}+Z_{BA}+Z_{CA}+aZ_{AB}+aZ_{BB}+aZ_{CB}+a^2Z_{AC}+a^2Z_{BC}+a^2Z_{CC}\right)/3\\
Z_{10} &= \left(Z_{AA}+aZ_{BA}+a^2Z_{CA}+Z_{AB}+aZ_{BB}+a^2Z_{CB}+Z_{AC}+aZ_{BC}+a^2Z_{CC}\right)/3\\
Z_{11} &= \left(Z_{AA}+aZ_{BA}+a^2Z_{CA}+a^2Z_{AB}+Z_{BB}+aZ_{CB}+aZ_{AC}+a^2Z_{BC}+Z_{CC}\right)/3\\
Z_{12} &= \left(Z_{AA}+aZ_{BA}+a^2Z_{CA}+aZ_{AB}+a^2Z_{BB}+Z_{CB}+a^2Z_{AC}+Z_{BC}+aZ_{CC}\right)/3\\
Z_{20} &= \left(Z_{AA}+a^2Z_{BA}+aZ_{CA}+Z_{AB}+a^2Z_{BB}+aZ_{CB}+Z_{AC}+a^2Z_{BC}+aZ_{CC}\right)/3\\
Z_{21} &= \left(Z_{AA}+a^2Z_{BA}+aZ_{CA}+a^2Z_{AB}+aZ_{BB}+Z_{CB}+aZ_{AC}+Z_{BC}+a^2Z_{CC}\right)/3\\
Z_{22} &= \left(Z_{AA}+a^2Z_{BA}+aZ_{CA}+aZ_{AB}+Z_{BB}+a^2Z_{CB}+a^2Z_{AC}+aZ_{BC}+Z_{CC}\right)/3
\end{aligned} \qquad (25)$$

Additional notes might be made on how this matrix works out for the typical transmission line.

In transmission lines, especially when phases are symmetrically spaced and phases are regularly transposed, it is frequently justified to assume the following symmetries in the $\mathbf{Z_{ABC}}$ impedance network:

$$\begin{aligned}
&X_{SELF}\left(X_S\right)=X_{AA}=X_{BB}=X_{CC}\\
&X_{MUTUAL}\left(X_M\right)=X_{AB}=X_{AC}=X_{BA}=X_{BC}=X_{CA}=X_{CB}\\
&X_{GROUND}\left(X_G\right)=X_{AG}=X_{BG}=X_{CG}\\
&R_{PHASE}\left(R_P\right)=R_A=R_B=R_C\\
&R_{GROUND}\left(R_G\right)=R_{AG}=R_{BG}=R_{CG}
\end{aligned} \qquad (26)$$

Substituting these impedances into (10) gives the following definitions of Z_S and Z_M and the various elements of $\mathbf{Z_{ABC}}$

$$\begin{aligned} Z_S &= Z_{AA} = Z_{BB} = Z_{CC} \\ &= R_P + R_G + j\left(X_S + X_G\right) \\ Z_M &= Z_{AB} = Z_{AC} = Z_{BA} = Z_{BC} = Z_{CA} = Z_{CB} \\ &= R_G + j\left(X_M + X_G\right) \end{aligned} \quad . \tag{27}$$

These simplifications give the $\mathbf{Z_{ABC}}$ matrix the form

$$\mathbf{Z_{ABC}} = \begin{bmatrix} Z_S & Z_M & Z_M \\ Z_M & Z_S & Z_M \\ Z_M & Z_M & Z_S \end{bmatrix} . \tag{28}$$

When we convert $\mathbf{Z_{ABC}}$ to $\mathbf{Z_{012}}$, we obtain the relatively simple diagonal matrix

$$\mathbf{Z_{012}} = \mathbf{A}^{-1} \begin{bmatrix} Z_S & Z_M & Z_M \\ Z_M & Z_S & Z_M \\ Z_S & Z_M & Z_S \end{bmatrix} \mathbf{A} = \begin{bmatrix} Z_S + 2Z_M & 0 & 0 \\ 0 & Z_S - Z_M & 0 \\ 0 & 0 & Z_S - Z_M \end{bmatrix} . \tag{29}$$

Hence, the 012 domain voltage drop equation becomes

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_S + 2Z_M & 0 & 0 \\ 0 & Z_S - Z_M & 0 \\ 0 & 0 & Z_S - Z_M \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} . \tag{30}$$

Equation (30) is typically stated as

$$\begin{bmatrix} V_{0,S} \\ V_{1,S} \\ V_{2,S} \end{bmatrix} - \begin{bmatrix} V_{0,R} \\ V_{1,R} \\ V_{2,R} \end{bmatrix} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} . \tag{31}$$

where:

$$\begin{aligned} Z_{00} = Z_0 &= Z_S + 2Z_M \\ &= \left(R_P + 3R_G\right) + j\left(X_S + 2X_M + 3X_G\right) \\ Z_{11} = Z_{22} = Z_1 = Z_2 &= Z_S - Z_M \\ &= R_P + j\left(X_S - X_M\right) \end{aligned} \tag{32}$$

Also note that via some algebraic manipulation of (32)

$$\begin{aligned} Z_S &= \left(Z_0 + 2Z_1\right)/3 \\ Z_M &= \left(Z_0 - Z_1\right)/3 \end{aligned} \quad . \tag{33}$$

4.a Calculating the Z_{ABC} Impedance Elements

We now need to determine the impedance elements in the $\mathbf{Z_{ABC}}$ (10) or $\mathbf{Z_{ABC+}}$ (15) matrices, and we need to determine how to calculate these impedances. We need to look forward a bit first. In the following sections we will show the calculations of two different types of reactances, X^C_{PP} and X^C_{PQ} where P and Q can be A, B, C, or N (or if needed, N_1, N_2, N_3, ...).

- X^C_{PQ}: If the subscripts differ from one another (e.g., X_{AB} or X_{CN}), the reference is to calculations seen in the next section, specifically in (36). In this path, current flows strictly in metallic paths such as phase conductors, overhead ground conductors, or quasi wires such as railroad tracks, but not the ground.
- X^C_{PP}: If the subscripts are the same (e.g., X_{AA} or X_{NN}), the reference is to the impedance of the indicated ground loop, calculated according to Carson's process, and final equation being (115). In this path, current flows out on the phase wire and returns as a diffuse ground current.
- R_P This refers to the AC resistance of the indicated phase conductor.
- R_{PG} This refers to the AC resistance of the indicated phase for the ground current loop calculated per Carson's equation, as seen in (115).

The impedances above are not entered directly into the impedance matrix, $\mathbf{Z_{ABC}}$ (10) or $\mathbf{Z_{ABC+}}$ (15) but a variation of this data is, according to the following equations.

Ground loops.
Assign half the flux in the loop to current in the phase conductor and half to current in the ground. R_{GP} is as calculated by Carson's equation (115), but subtract out the known phase conductor resistance.

$$\begin{array}{lll} X_{AA} = 0.5X^C_{AA} & X_{AG} = 0.5X^C_{AA} & R_{AG} = R^C_{AA} - R_A \\ X_{BB} = 0.5X^C_{BB} & X_{BG} = 0.5X^C_{BB} & R_{BG} = R^C_{BB} - R_B \\ X_{CC} = 0.5X^C_{CC} & X_{CG} = 0.5X^C_{CC} & R_{CG} = R^C_{CC} - R_C \\ X_{NN} = 0.5X^C_{NN} & X_{NG} = 0.5X^C_{NN} & R_{NG} = R^C_{NN} - R_N \end{array} \tag{34}$$

Mutual Impedances.
If one wishes to verify these equations, envision the impedance that will be seen in the $\mathbf{Z_{ABC}}$ impedance network for a 3 phase, phase to phase, and SLG fault. Insert the equations below into $\mathbf{Z_{ABC}}$, and one should find appropriate impedances for the indicated fault type.

$$\begin{array}{lll} X_{AB} = 0.5X^C_{AA} - X^C_{AB} & X_{BA} = 0.5X^C_{BB} - X^C_{BA} & X_{CA} = 0.5X^C_{CC} - X^C_{CA} \\ X_{AC} = 0.5X^C_{AA} - X^C_{AC} & X_{BC} = 0.5X^C_{BB} - X^C_{BC} & X_{CB} = 0.5X^C_{CC} - X^C_{CB} \\ X_{AN} = 0.5X^C_{AA} - X^C_{AN} & X_{BN} = 0.5X^C_{BB} - X^C_{BN} & X_{CN} = 0.5X^C_{CC} - X^C_{CN} \end{array} \tag{35}$$

4.b Impedance With all Current in Metallic Conductor

The easy impedance calculation is for current out on one conductor and back on another conductor. Most textbooks in the power engineering field have good sections on the impedance for this condition, especially when the conductors are isolated from ground. We will not review the logic behind those equations, but let us review a few important points. For spacing D_{AB} between the A and B phase, and effective conductor radius (for inductance calculations) of conductor A of radius r_A

$$X_{AB}^{C} = \frac{\omega\mu}{2\pi} ln \frac{D_{AB}}{r_A} \quad \Omega/m \tag{36}$$

Note this is the effective radius, not the actual radius; see [3] and [4] for the calculation of a conductor effective radius.

Note the process we are using does not use the GMD between the conductors. The impedance of each respective wire is used.

Hence, given spacing of all the conductors, we can rapidly fill in the impedances in our matrix. However, when we get to the self impedances with earth return, we face a hurdle. Not many resources cover the topic well, and many revert to the "Carson's Equations." [1]. These equations are not easy to work with and have a "Gee, I wonder where that came from" nature. We will proceed to analyze Carson's old work to see if we can shed some light on the topic.

4.c The Ground Current Loop and The Development of Carson's Equations

To find the impedance of the ground loop, the most common reference is [1]. Rather than simply provide equations from that work, we will attempt to develop the paper's concepts step by step. To follow, one will need to be aware of electromagnetic wave propagation. Basic electromagnetic theory is reviewed in the Appendix, and [5] is a good and inexpensive reference that was used in some of the development of the appendix. Also, as previously mentioned, the analysis below would not have reached nearly as far without the insights from [2].

The System for Analysis

The right hand Cartesian system to be analyzed is seen in Figure 3. A voltage source is driving current in the $\hat{z}$ direction down a line, with earth return. The line is at height h on the $\hat{y}$ axis.

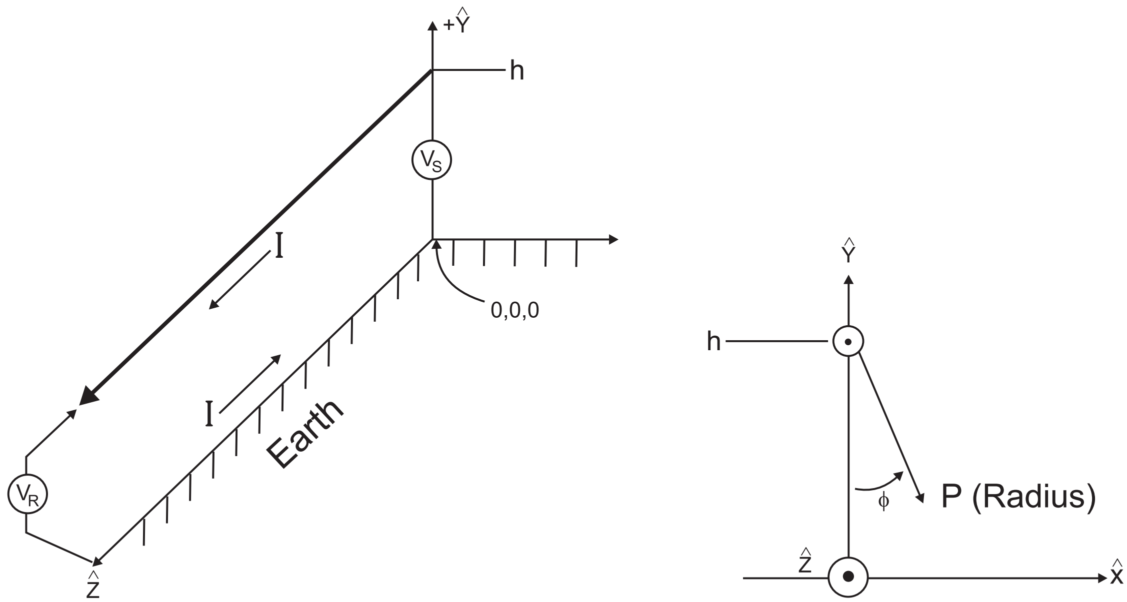

Figure 3 - Coordinate System

The assumed electric fields are shown in Figure 4. Note that in Figure 4 the **E** field in the ground is only in the $\hat{z}$ direction. In the air, there is a large E field component in the $\hat{x}$ and $\hat{y}$ direction, but there is a $\hat{z}$ component as well.

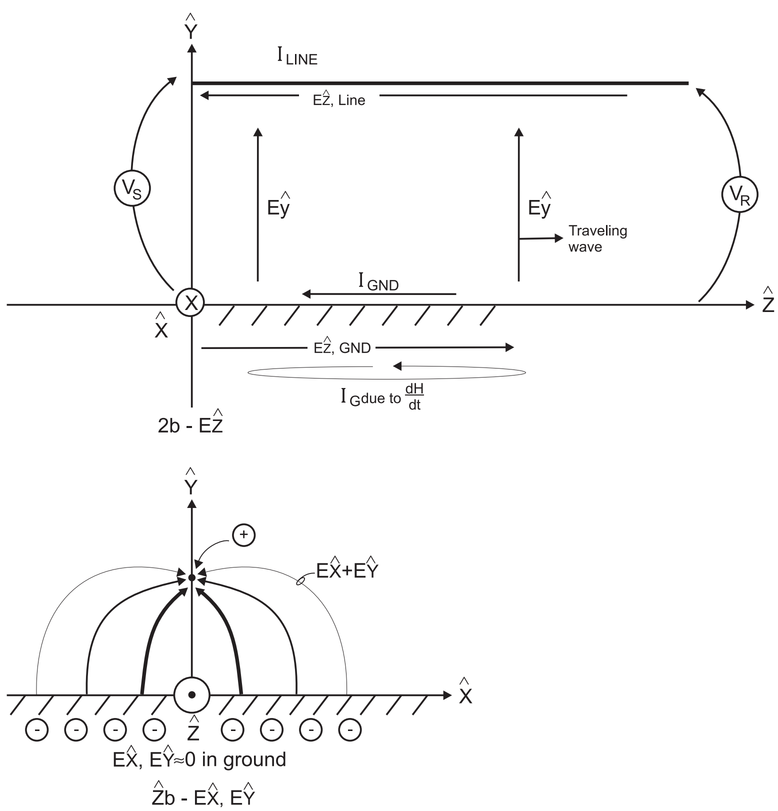

Figure 4 - E Field

The difficulties in analyzing voltage drop can be seen in the lack of definition of the ground current distribution and the **H** field, seen in Figures 5 and 6. First, the **H** field in the ground due to line current is distorted by eddy currents in the ground, such that the **H** field ground penetration is reduced relative to penetration of air. Next, since the current in the ground is distributed in an unclear pattern, the **H** field in the ground and air that is due to ground currents is not directly known. Until we determine how currents and the **H** field are distributed, we cannot calculate the inductance of the ground current loop.

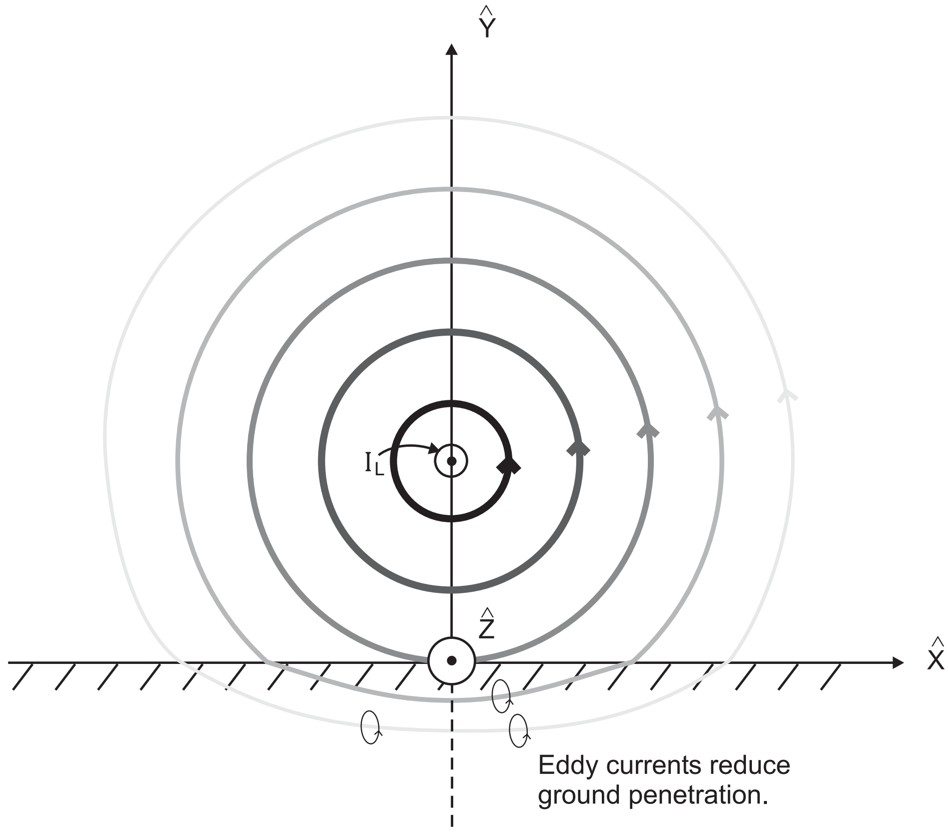

Figure 5 - H Field Due to Line Current

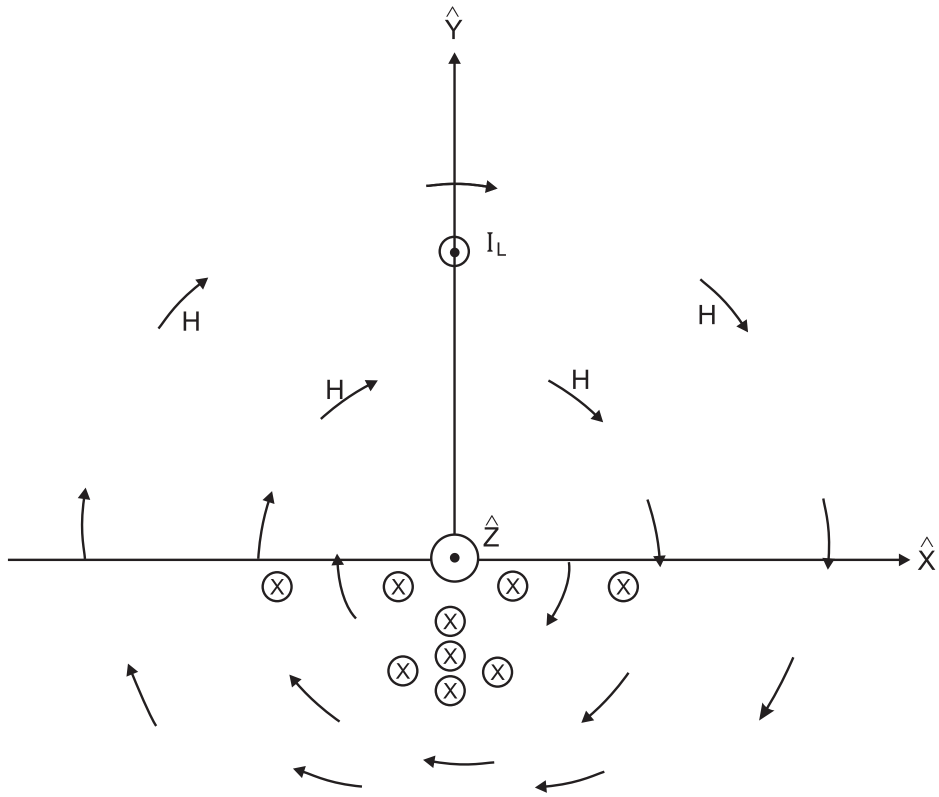

Figure 6 - H Field Due to Ground Current

Notation Used for Fields

- The format of the equation for **E** and **H** fields in the Cartesian system, and allowing for the fields to vary over x, y, z, and t, are

$$\mathbf{E} = \mathbf{E}_x(x,y,z,t)\hat{x} + \mathbf{E}_y(x,y,z,t)\hat{y} + \mathbf{E}_z(x,y,z,t)\hat{z} \quad (37)$$

$$\mathbf{H} = \mathbf{H}_x(x,y,z,t)\hat{x} + \mathbf{H}_y(x,y,z,t)\hat{y} + \mathbf{H}_z(x,y,z,t)\hat{z}. \quad (38)$$

  which for simplicity may be shown as

$$\mathbf{E} = \mathbf{E}_x\hat{x} + \mathbf{E}_y\hat{y} + \mathbf{E}_z\hat{z} \quad (39)$$

$$\mathbf{H} = \mathbf{H}_x\hat{x} + \mathbf{H}_y\hat{y} + \mathbf{H}_z\hat{z}. \quad (40)$$

- If all variations with respect to time are a single frequency time harmonic, exponential terms are normally used rather than sine and cosine terms. Exponential notation is further discussed in Appendix Section C.1. The field equations for time harmonic conditions have the form of

$$\mathbf{E} = \left(\mathbf{E}_x\hat{x} + \mathbf{E}_y\hat{y} + \mathbf{E}_z\hat{z}\right)e^{j(\omega t+\varphi)} \quad (41)$$

$$\mathbf{H} = \left(\mathbf{H}_x\hat{x} + \mathbf{H}_y\hat{y} + \mathbf{H}_z\hat{z}\right)e^{j(\omega t+\varphi)}. \quad (42)$$

- For many cases, the time function is implied and not explicitly stated. e.g., one may need to mentally add the $e^{j\omega t-\gamma z}$ term. Throughout, Carson allows this term to be assumed but, in some cases, this paper includes the term for completeness. For simplicity, the paper does not include the φ offset in the $e^{j\omega t}$ term seen above.

- In this paper, the lack of a dimension subscript (e.g., x, y, or z) for **E** and **H** indicates the reference is to the complete composite field, consisting of **E** or **H** in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ direction.
- When **E** or **H** have the subscript x, y, or z, the equation for **E** or **H** in the indicated direction is implied. For instance: $\mathbf{H}_X$ implies a function $\mathbf{H}_x(x,y,z,t)\hat{x}$ where $\mathbf{H}_x\hat{x}$ varies with x, y, z, and t. However, the symbol "$\hat{x}$" is not part of the equation; it is used only to give direction in space to the results of the equation.
- When a field or function is followed by dimensions in parenthesis, such as $\mathbf{H}_x$(x, y), this is meant to point to the concept that the field is considered only in terms of the indicated variables; e.g., for $\mathbf{H}_x$(x, y) the **H** field's $\hat{x}$ vector only changes with x and y, and changes in z or t have no effect on the equation. However, $\mathbf{H}_X$ by itself will tend to imply that the equation might be a function of x, y, z, and t.
- Many equations to follow will look somewhat different than the equations in Carson's work. The reasons include:
  a. This paper uses of SI-MKS system rather than Carson's CGS system, See Appendix Section A for a discussion of CGS unit systems. Of particular note, any equation herein that relates **I** to **H** will have a 4π factor missing in Carson's equations, and any equation herein that relate **H** to **E** will have an extra μ factor not found in Carson's equations. Also, Carson's system had a permeability constant of 1, so the permeability constant did not appear in his equations. One should read the comments associated with Appendix, table 4, especially (A.18).
  b. This paper includes the $e^{j\omega t-\gamma z}$ propagation term in some equations where Carson left the term as implied.
  c. This paper uses η rather than Carson's μ for a integration variable.
  d. This paper uses complex exponential terms to indicate sine and cosine functions, but Carson used sinusoidal functions.

Simplifying Assumptions

The equations will be different for the field in the air and the field in the ground, and each media will have different simplifications to the field equations. To simplify the equations, assume that:

- The end effects are not considered in the paper; i.e., the modeling is only on the mid-span portion of a relatively long line with no discontinuities.
- The E-M wave in the $\hat{z}$ direction is described by a traveling wave describe in the format of $e^{j\pi t-\gamma z}$. See Appendix (A.60).
- In the ground $\mathbf{E}_x\hat{x}$ and $\mathbf{E}_y\hat{y}$ are negligible, and the only E field is in the $\hat{z}$ direction. The **E** field in the ground exists because d**H**/dt fields in the ground induce voltage in the $\hat{z}$ direction.
- In both the air and the ground, $\mathbf{H}_z\hat{z}$ can be considered negligible.

Applying these concepts, the equations for **E** and **H** in the air (y ≥ 0) take on the format of

$$\mathbf{E}_A = \left(\mathbf{E}_{xA}(x,y)\hat{x} + \mathbf{E}_{yA}(x,y)\hat{y} + \mathbf{E}_{zA}(x,y)\hat{z}\right)e^{j\omega t-\gamma z} \tag{43}$$

$$\mathbf{H}_A = \left(\mathbf{H}_{xA}(x,y)\hat{x} + \mathbf{H}_{yA}(x,y)\hat{y}\right)e^{j\omega t-\gamma z}. \tag{44}$$

In the ground (y ≤ 0) they take on the format

$$\mathbf{E}_G = \left(\mathbf{E}_{zG}(x,y)\hat{z}\right)e^{j\omega t-\gamma z} \tag{45}$$

$$\mathbf{H}_G = \left(\mathbf{H}_{xG}(x,y)\hat{x} + \mathbf{H}_{yG}(x,y)\hat{y}\right)e^{j\omega t-\gamma z}. \tag{46}$$

At y=0, both equations will be considered valid. This will constitute the boundary condition that will help determine the nature of the equations.

The propagation constant γ is reviewed in the Appendix, particularly in the equations leading to (A.53). It has the form of

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)\,. \tag{47}$$

In air, σ = 0, but in ground σ is large, and in any conductive media of interest, σ>>ωε, so in air and ground, the equation reduces to:

In Air: $\gamma_A^2 = -\omega^2\mu\varepsilon$

In Ground: $\gamma_G^2 \approx j\omega\mu\sigma$ (48)

In ground, since $\sigma \gg \omega\varepsilon$, then $\gamma_G \gg \gamma_A$

See further discussions about propagation constants in Appendix Section C2, C3, and C4, particularly the material leading to equation (A.75).

Equations for H in the Ground and Air

The analysis process will be to first find equations to describe the **H** field in the ground and in the air. Once these are described, we will remove some unknowns from the equations by making the air and ground equations at the boundary between ground and air.

H Field in the Ground

To find the $\mathbf{H}_G$ we start by describing $\mathbf{E}_G$, which is generated by $\mathbf{H}_G$. The assumptions above simplify the **E** wave equation seen in (A.30), and (A.51). If the assumption is that a value is negligible, then for at least the frequencies of interests in power system analysis, the 2nd derivative would also be negligible. Dropping the appropriate terms in (A.30), the equation reduces to

$$\frac{\partial^2 \mathbf{E}_{z,G}}{\partial x^2} + \frac{\partial^2 \mathbf{E}_{z,G}}{\partial y^2} + \frac{\partial^2 \mathbf{E}_{z,G}}{\partial z^2} = \gamma^2 \mathbf{E}_{z,G}\hat{z}\,. \tag{49}$$

The γ propagation constant on the right side is referring to the propagation constant of the electromagnetic field in the ground (γ_G). However, the $\frac{\partial^2 \mathbf{E}_{z,G}}{\partial z^2}$ term on the left side is the voltage in the ground that is induced from the **H** field due to line and ground currents. These currents are propagating mainly in air, and approximately follow the concepts of Appendix Section C.4, and have a small propagation constant (γ_L) relative to (γ_G). We can simplify (49) by applying $\frac{\partial^2 \mathbf{E}_{z,G}}{\partial z^2} = \gamma_L^2 \mathbf{E}_{z,G}(z)$, which gives us

$$\frac{\partial^2 \mathbf{E}_{z,G}}{\partial x^2} + \frac{\partial^2 \mathbf{E}_{z,G}}{\partial y^2} + \gamma_L^2 \mathbf{E}_{z,G}(z) = \gamma_G^2 \mathbf{E}_{z,G}\,. \tag{50}$$

We will assume that $\gamma_G \gg \gamma_L$ and since from (48), $\gamma_G^2 \approx j\omega\mu\sigma$, with little error we can restate (50) as

$$\frac{\partial^2 \mathbf{E}_{z,G}}{\partial x^2} + \frac{\partial^2 \mathbf{E}_{z,G}}{\partial y^2} = j\omega\mu\sigma \mathbf{E}_{z,G}. \tag{51}$$

Now assume that $\mathbf{E}_{z,G}$ can be stated as two separate equations, each independent functions of x and y

$$\mathbf{E}_{z,G} = \mathbf{E}_{z,G}(x)\,\mathbf{E}_{z,G}(y). \tag{52}$$

This allows (51) to be restated as

$$\mathbf{E}_{z,G}(y)\frac{\partial^2 \mathbf{E}_{z,G}(x)}{\partial x^2} + \mathbf{E}_{z,G}(x)\frac{\partial^2 \mathbf{E}_{z,G}(y)}{\partial y^2} = j\omega\mu\sigma\left(\mathbf{E}_{z,G}(x)\;\; \mathbf{E}_{z,G}(y)\right). \tag{53}$$

Which can be restated as

$$\frac{1}{\mathbf{E}_{z,G}(x)}\frac{\partial^2 \mathbf{E}_{z,G}(x)}{\partial x^2} + \frac{1}{\mathbf{E}_{z,G}(y)}\frac{\partial^2 \mathbf{E}_{z,G}(y)}{\partial y^2} = j\omega\mu\sigma + \mathbf{H}_{y,A}^{G} + \eta^2 - \eta^2. \tag{54}$$

The "$+\eta^2$-η^2" factor in (54) is adding 0 to the right hand side, so it has not changed the equation. It was introduced to allow us to separated the variables x and y. The value for η^2 can take on any value. Note: we do not know the correct value for η; any value will fit the needs of the equation. In the equations ahead, we will make the assumption that η takes on every possible value.

(Side note: Carson used the symbol μ where this paper uses η. The change from μ to η was made to avoid confusion with μ as the permeability constant.)

We can determine that (54) can be seen as the sum of

$$\frac{1}{\mathbf{E}_{z,G}(x)}\frac{\partial^2 \mathbf{E}_{z,G}(x)}{\partial x^2} = -\eta^2 \tag{55}$$

$$\frac{1}{\mathbf{E}_{z,G}(y)}\frac{\partial^2 \mathbf{E}_{z,G}(y)}{\partial y^2} = j\omega\mu\sigma + \eta^2. \tag{56}$$

Rearranging (55) and (56) we have

$$\frac{\partial^2 \mathbf{E}_{z,G}(x)}{\partial x^2} = -\eta^2 \mathbf{E}_{z,G}(x) \tag{57}$$

$$\frac{\partial^2 \mathbf{E}_{z,G}(y)}{\partial y^2} = \left(\eta^2 + j\omega\mu\sigma\right)\mathbf{E}_{z,G}(y). \tag{58}$$

Similar to the analysis of electromagnetic wave propagation in Appendix Section C.2 to C.4, the format of the equations that matches the above equations is f(x) = Ae^{bx}. The equations are

$$\mathbf{E}_{z,G}(x) = F_G(\eta)e^{j\eta x} \tag{59}$$

$$\mathbf{E}_{z,G}(y) = F_G(\eta)e^{y\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} . \tag{60}$$

In the analysis ahead, we will develop an equation for F_G that incorporates line current. In accordance with (52), the net equation for $\mathbf{E}_{z,G}$ is the multiplication of these last two equations. Also, we will assume that there is only one F_G equation. Lastly, we need to introduce a “-1” factor to account for the concept that positive $\mathbf{E}_z$ is looking back toward Z=0, in the reverse direction of $+\hat{z}$. Hence, for a given η,

$$\mathbf{E}_{z,G}(\eta, x, y) = -F_G(\eta)e^{jux}e^{y\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} \quad \hat{z} . \tag{61}$$

Carson’s solution for removing η from the equation was to assume that η takes on all values; hence, proceeded with the concept that $\mathbf{E}_{z,G}$ is an integration of (61) over all η. If this is the case, the equation for $\mathbf{E}_{z,G}$ takes on the form of

$$\mathbf{E}_{z,G} = -e^{j\omega t - \gamma z}\int_0^\infty F_G(\eta)e^{j\eta x}e^{y\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} d\eta \quad \hat{z} . \tag{62}$$

Carson (1)

Note the f(t,z) traveling wave function has been shown for completeness.

As evidence of how sparingly Carson’s paper reported his thought processes, equation (62) is equation (1) in Carson’s paper. Carson simply states this equation as a truth that needs little explanation. As the paper progresses, it makes a number of large steps in logic with minimal explanation. Maybe Carson had extensive notes on this paper’s topic, but was restricted in how much he could put into print.

The $\mathbf{H}_G$ equations can be derived from the $\mathbf{E_G}$ equations above. First, since $\mathbf{E}_x\hat{x} = \mathbf{E}_y\hat{y} = 0$ in our analysis, then $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$, as interpreted by (A.26) and (A.11), becomes

$$\frac{\partial \mathbf{E}_{z,G}}{\partial y} = -j\omega\mu\mathbf{H}_{x,G} \quad \hat{x} \tag{63}$$

$$\frac{\partial \mathbf{E}_{z,G}}{\partial x} = j\omega\mu\mathbf{H}_{y,G} \quad \hat{y} . \tag{64}$$

Applying the above equations to $\mathbf{E}_{z,G}$ as described in (62), and solving for $\mathbf{H}_G$ gives

$$\mathbf{H}_{x,G} = \frac{e^{j\omega t - \gamma z}}{j\omega\mu}\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}\int_0^\infty F_G(\eta)e^{j\eta x}e^{y\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} d\eta \quad \hat{x} \tag{65}$$

$$\mathbf{H}_{y,G} = \frac{e^{j\omega t - \gamma z}}{j\omega\mu} \eta \int_0^{\infty} F_G(\eta) e^{j(\eta x - \pi/2)} e^{y\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} d\eta \quad \hat{y}. \tag{66}$$

Carson (2),(3)

Note:

- $\mathbf{H}_G$ field in the ground is, in this analysis, strictly in the $\hat{x}$ and $\hat{y}$ direction, and has no $\hat{z}$ direction, though the magnitude in the $\hat{x}$ and $\hat{y}$ direction are affected by the $\hat{z}$ traveling wave effect.
- There is no direct inclusion of current (or voltage) in the above equations for **H**. Current will appear in the F_G factors that will be defined later when boundary conditions are analyzed.
- The use of j, -1, and $e^{j\pi/2}$ terms as seen above was arranged in the manner shown for eventual simplifications seen in (99) through (102).

H Field in the Air

Just as in the ground, the $\mathbf{H}\hat{z}$ field in the air, in this analysis, is negligible; hence, **H** is strictly in the $\hat{x}$ and $\hat{y}$ direction, though the magnitude in the $\hat{x}$ and $\hat{y}$ direction are affected by the z traveling wave effect. In the air, there is an **H** field due to ground currents and line currents which sum together for net field in the air:

$$\mathbf{H}_{x,A} = \mathbf{H}^{L}_{x,A} + \mathbf{H}^{G}_{x,A} \tag{67}$$

$$\mathbf{H}_{y,A} = \mathbf{H}^{L}_{y,A} + \mathbf{H}^{G}_{y,A} \tag{68}$$

***H_A* due to Line Currents**

The $\mathbf{H}_A$ field due to the line currents follows the classic concept that the line integral of the **H** field in a circle around a conductor is equal to the enclosed current (A.6). The resultant equations are:

$$\mathbf{H}^{L}_{x,A} = \frac{cos(\Phi)}{2\pi p} \mathbf{I}_L \quad \hat{x} \tag{69}$$

$$\mathbf{H}^{L}_{y,A} = \frac{sin(\Phi)}{2\pi p} \mathbf{I}_L \quad \hat{y} \tag{70}$$

Carson (4), (5)

where

$$p = \left(x^2 + (y-h)^2\right)^{0.5} \quad \text{which at y=0 is:} \quad p = \left(x^2 + h^2\right)^{0.5}. \tag{71}$$

If one compares to Carson's version of these equations, one will see a 4π factor difference. One should see that this arises out of Carson's use of a CGS system. Compare the SI-MKS (A.6) to the equivalent CGS (A.18). The CGS system has units for **H** vs. **I** that differ from the SI-MKS system by a 4π factor.

Late, when we determine $F_G(u)$ and $F_A(u)$ by applying the boundary condition of y=0, we will need to express the value of (69) and (70) in a format the same as (65) and (66), above, and (95) and (96) below. Consider the following format:

$$\mathbf{H}^{L}_{x,A} = \frac{I_L}{2\pi}\int_0^{\infty} e^{-h\eta + j\eta x} d\eta \quad \hat{x} \tag{72}$$

$$\mathbf{H}^{L}_{y,A} = \frac{I_L}{2\pi}\int_0^{\infty} e^{-h\eta + j(\eta x - \pi/2)} d\eta \quad \hat{y}\,. \tag{73}$$

Carson (9, 10)

Also, it might be noted that (72) and (73) are also modulated by the same traveling wave $e^{j\omega t-\gamma z}$ factor seen in (65) and (66).

If one integrates these equations, they reduce to (69) and (70), indicating that they are valid restatements of (69) and (70). For instance integrating (73) gives:

$$\mathbf{H}^{L}_{y,A} = Re\left(\frac{I_L}{2\pi}\frac{e^{-h\eta + j(\eta x - \pi/2)}}{(-h + jx)}\Bigg|_{\eta=0}^{\eta=\infty}\right) \quad \hat{y}$$

$$= Re\left(\frac{I_L}{2\pi}\left(0 - \frac{e^{-j\pi/2}(-h - jx)}{p \cdot p}\right)\right) = \frac{I_L}{2\pi}\left(\frac{x}{p \cdot p}\right) = \frac{I_L}{2\pi}\left(\frac{\sin\Phi}{p}\right) \quad \hat{y}.$$

***H_A* due to Ground Currents**

We are going to go through a process very similar to (49) through (62). First, examine the **H** wave equation (A.31). If $\mathbf{H}\,\hat{z}$ is a negligible component, $\partial^2\mathbf{H}/\partial t^2$ is also negligible; hence, the equation reduces to

$$\left(\frac{\partial^2 \mathbf{H}^{G}_{x,A}}{\partial x^2} + \frac{\partial^2 \mathbf{H}^{G}_{x,A}}{\partial y^2}\right) = \gamma^2 \mathbf{H}^{G}_{x,A} \quad \hat{x} \tag{74}$$

$$\left(\frac{\partial^2 \mathbf{H}^{G}_{y,A}}{\partial x^2} + \frac{\partial^2 \mathbf{H}^{G}_{y,A}}{\partial y^2}\right) = \gamma^2 \mathbf{H}^{G}_{y,A} \quad \hat{y}\,. \tag{75}$$

Assume that $\mathbf{H}_{x,A}$ and $\mathbf{H}_{y,A}$ can be stated as two separate equations that are functions of x and y. The equations have the form of

$$\mathbf{H}^{G}_{x,A} = \mathbf{H}^{G}_{x,A}(x)\,\mathbf{H}^{G}_{x,A}(y) \tag{76}$$

$$\mathbf{H}^{G}_{y,A} = \mathbf{H}^{G}_{y,A}(x)\,\mathbf{H}^{G}_{y,A}(y)\,. \tag{77}$$

This allows (74) and (75) to be restated as

$$\frac{1}{\mathbf{H}^{G}_{x,A}(x)}\frac{\partial^2 \mathbf{H}^{G}_{x,A}(x)}{\partial x^2} + \frac{1}{\mathbf{H}^{G}_{x,A}(y)}\frac{\partial^2 \mathbf{H}^{G}_{x,A}(y)}{\partial y^2} = -\omega^2\mu\varepsilon + \eta^2 - \eta^2 \tag{78}$$

$$\frac{1}{\mathbf{H}^{G}_{y,A}(x)}\frac{\partial^2 \mathbf{H}^{G}_{y,A}(x)}{\partial x^2} + \frac{1}{\mathbf{H}^{G}_{y,A}(y)}\frac{\partial^2 \mathbf{H}^{G}_{y,A}(y)}{\partial y^2} = -\omega^2\mu\varepsilon + \eta^2 - \eta^2\,. \tag{79}$$

Again, the "$+\eta^2-\eta^2$" factor was reintroduced to allow us to separate the variables x and y. We can simplify the right side, assuming that $\omega^2\mu\varepsilon$ is a very small number, especially for the frequencies of interest in power system analysis, so that

$$-\omega^2\mu\varepsilon+\eta^2 \approx \eta^2 . \tag{80}$$

Separating the variables in the same process as for $\mathbf{E}_{z,G}$ gives

$$\mathbf{H}^G_{x,A}: \quad \frac{1}{\mathbf{H}^G_{x,A}(x)}\frac{\partial^2\mathbf{H}^G_{x,A}(x)}{\partial x^2}=-\eta^2 \quad \text{and} \quad \frac{1}{\mathbf{H}^G_{x,A}(y)}\frac{\partial^2\mathbf{H}^G_{x,A}(y)}{\partial y^2}=\eta^2 \qquad (81)\ (82)$$

$$\mathbf{H}^G_{y,A}: \quad \frac{1}{\mathbf{H}^G_{y,A}(x)}\frac{\partial^2\mathbf{H}^G_{y,A}(x)}{\partial x^2}=-\eta^2 \quad \text{and} \quad \frac{1}{\mathbf{H}^G_{y,A}(y)}\frac{\partial^2\mathbf{H}^G_{y,A}(y)}{\partial y^2}=\eta^2 . \qquad (83)\ (84)$$

The result is that we have two sets of differential equations:

$$\mathbf{H}^G_{x,A}: \quad \frac{\partial^2\mathbf{H}^G_{x,A}(x)}{\partial x^2}=-\eta^2\mathbf{H}^G_{x,A}(x) \quad \text{and} \quad \frac{\partial^2\mathbf{H}^G_{x,A}(y)}{\partial y^2}=\eta^2\mathbf{H}^G_{x,A}(y) \qquad (85)\ (86)$$

$$\mathbf{H}^G_{y,A}: \quad \frac{\partial^2\mathbf{H}^G_{y,A}(x)}{\partial x^2}=-\eta^2\mathbf{H}^G_{y,A}(x) \quad \text{and} \quad \frac{\partial^2\mathbf{H}^G_{y,A}(y)}{\partial y^2}=\eta^2\mathbf{H}^G_{y,A}(y) . \qquad (87)\ (88)$$

that give us these four equations:

$$\mathbf{H}^G_{x,A}(x)=F_A(\eta)e^{j\eta x} \quad \text{and} \quad \mathbf{H}^G_{y,A}(y)=F_A(\eta)e^{-\eta y} \qquad (89)\ (90)$$

$$\mathbf{H}^G_{y,A}(x)=F_A(\eta)e^{j(\eta x+\pi/2)} \quad \text{and} \quad \mathbf{H}^G_{y,A}(y)=F_A(\eta)e^{-\eta y} . \qquad (91)\ (92)$$

The assignments of $+\eta^2$ to (82) and (84) indicates an **H** field that decays exponentially with regard to y. The "+π/2" constant in (91) might be understood by inspecting Figure 4. In Figure 4, for positive x, the $\hat{y}$ component of the **H** field is negative, and should be 0 at x=0. Note that by adding +π/2 to the complex exponential in (91), when the real component of the term is taken, we have "-sin(ηx)." This agrees with "at x = 0, function is 0, and for positive x, the value is negative."

In the equations to come, we will develop an equation for F_A that incorporates line current. In accordance with (76) and (77) the net equation for $\mathbf{H}_{x,A}$ and $\mathbf{H}_{y,A}$ is the multiplication of (89)x(90) and (91)x(92). Further, we will simplify and state that there is only one F_A equation. Hence, for a given η,

$$\mathbf{H}^G_{x,A}(\eta,x,y)=F_A(\eta)e^{j\eta x}e^{-\eta y} \tag{93}$$

$$\mathbf{H}^G_{y,A}(\eta,x,y)=F_A(\eta)e^{j(\eta x-\pi/2)}e^{-\eta y} . \tag{94}$$

Using the same logic as for (62), assume that the net equation is the combination of all possible η, and re-introducing the $\hat{z}$ traveling wave and time function

$$\mathbf{H}^{G}_{x,A} = e^{j\omega t-\gamma z}\int_0^\infty F_A(\eta)e^{j\eta x}e^{-\eta y}d\eta \quad \hat{x} \tag{95}$$

$$\mathbf{H}^{G}_{y,A} = -e^{j\omega t-\gamma z}\int_0^\infty F_A(\eta)e^{j(\eta x-\pi/2)}e^{-\eta y}d\eta \quad \hat{y} \tag{96}$$

Carson (7, 8)

Again, we still have considerable work ahead in that we do not know what constitutes $F_A(u)$.

Applying the Boundary Conditions, H at y = 0

In this step, to fully define the $\mathbf{H}_G$ and $\mathbf{H}_A$, we apply the concept that $\mathbf{H}_G = \mathbf{H}_A$ at the ground/air boundary. At the interface between the ground and the air, the $\hat{x}$ and $\hat{y}$ components of the **H** field are each continuous. Hence

$$L = \frac{\mu}{\pi}\mathit{ln}\frac{D}{r_e} \quad \text{H/m} \tag{97}$$

$$\mathbf{H}_{x,G} = \mathbf{H}^{L}_{x,A} + \mathbf{H}^{G}_{x,A}\,. \tag{98}$$

For the $\hat{x}$ equations, we apply (65), (72), and (95), to obtain

$$\frac{\left(u^2+j\omega\mu\sigma\right)^{0.5}}{j\omega\mu}\int_0^\infty F_G(\eta)e^{j\eta x}e^{y\left(\eta^2+j\omega\mu\sigma\right)^{0.5}}d\eta = \frac{I_L}{2\pi}\int_0^\infty e^{-h\eta+jx\eta}d\eta + \int_0^\infty F_A(\eta)e^{j\eta x}e^{-\eta y}d\eta\,. \tag{99}$$

For the $\hat{y}$ equations, we apply (66), (73), and (96) to obtain

$$\frac{\eta}{j\omega\mu}\int_0^\infty F_G(\eta)e^{j(\eta x-\pi/2)}e^{y\left(\eta^2+j\omega\mu\sigma\right)^{0.5}}d\eta = \frac{I_L}{2\pi}\int_0^\infty e^{-h\eta+j(x\eta-\pi/2)}d\eta - \int_0^\infty F_A(\eta)e^{j(\eta x-\pi/2)}e^{-\eta y}d\eta\,. \tag{100}$$

We want to evaluate the above equations at y = 0. Further, we need to remove the integration temporarily. If the two sides of the equations are equal for any and every single given η, then the integrals are equal, so we can work without the integrals. Hence, setting y = 0 and selecting an arbitrary η and canceling a common $e^{j\eta x}$ and $e^{j\eta x-\pi/2}$ terms, the equations reduce to

$$\frac{\left(\eta^2+j\omega\mu\sigma\right)^{0.5}}{j\omega\mu}F_G(\eta) = \frac{I_L}{2\pi}e^{-h\eta} + F_A(\eta) \tag{101}$$

$$\frac{\eta}{j\omega\mu}F_G(\eta) = \frac{I_L}{2\pi}e^{-h\eta} - F_A(\eta)\,. \tag{102}$$

We can resolve the above two equations into equations for $F_G(u)$ and $F_A(u)$,

$$F_G(\eta) = \left(\frac{j\omega\mu}{\left(\eta^2 + j\omega\mu\sigma\right)^{0.5} + \eta} \right) \frac{e^{-h\eta}}{\pi} I_L \tag{103}$$

$$F_A(u) = \left(\frac{\left(\eta^2 + j\omega\mu\sigma\right)^{0.5} - \eta}{\left(\eta^2 + j\omega\mu\sigma\right)^{0.5} + \eta} \right) \frac{e^{-h\eta}}{2\pi} I_L . \tag{104}$$

Carson (11,12)

Revised Equation for Voltage in the Ground
Substituting (103) in the equation for $\mathbf{E}_{z,G}$ (62) gives

$$\mathbf{E}_{z,G} = -e^{j\omega t - \gamma z} \left(\frac{j\omega\mu}{\pi} \right) I_L \int_0^\infty \left(\frac{1}{\left(\eta^2 + j\omega\mu\sigma\right)^{0.5} + \eta} \right) e^{-h\eta} e^{j\eta x} e^{y\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} d\eta \quad \hat{z} . \tag{105}$$

Carson (13)

The equation above is a bit complicated, so Carson introduced a substitution of variables. His approach is to let

$$\begin{aligned} K &= (\omega\mu\sigma)^{0.5} \\ x' &= Kx \\ y' &= Ky \\ h' &= Kh \end{aligned} \tag{106}$$

and then apply a variable substitution, so that (105) becomes

$$\mathbf{E}_{z,G} = -e^{j\omega t - \gamma z} \left(\frac{\omega\mu}{\pi} \right) I_L \int_0^\infty \left(\left(\eta^2 + j\right)^{0.5} - \eta \right) e^{-h'\eta} e^{j\eta x'} e^{y'\left(\eta^2 + j\omega\mu\sigma\right)^{0.5}} d\eta \quad \hat{z} . \tag{107}$$

Carson (14)

A few clues for tracing how (105) becomes (107):

- For a given η, the change of variables acts to increase the exponential parts by a factor of K. To keep the values in each exponent the same as before the change, the remainder of the exponent has to be multiplied by 1/K. This implies that another change of variables has occurred in each exponential term: η' = η/K.
- The effect of using a change of variables for η is that it reduces the integration by 1/K so the overall integration has to be multiplied by K to compensate. (This is one factor I had to think about quite a bit; if a reader has comments, please contact me. If this effect does not occur, a K factor in the next bullet is unaccounted for.)
- In the "$\left(\eta^2 + j\omega\mu\sigma\right)^{0.5} + \eta$" portion of (105) that did not have an exponential term, a K multiplier needs to be added in front of each η to bring it back to the value before the change of variables. One might think through this conversion process:

$$\left(\frac{1}{\left(\eta^2+jK^2\right)^{0.5}+\eta}\right)\Rightarrow\left(\frac{1}{\left(K^2\eta'^2+jK^2\right)^{0.5}+K\eta'}\right)\left(\frac{\left(K^2\eta'^2+jK^2\right)^{0.5}-K\eta'}{\left(K^2\eta'^2+jK^2\right)^{0.5}-K\eta'}\right)=\frac{(\eta'+j)^{0.5}-\eta}{jK}.$$

Voltage in the Air

At this point Carson makes some leaps in logic that this author (JJH) has only partially comprehended. If a reader has a firm grasp on the concepts to follow, I'd appreciate your contact.

Carson reports a measure of $\mathbf{E}_Z$ in the air, at point x, y, relative to point x, 0 (i.e., ground)

$$\mathbf{E}_{z,A}(x,y)-\mathbf{E}_{z,A}(x,0)=-j\omega\mu\int_0^y \mathbf{H}_x(x,y)\,dy-\frac{\partial \mathbf{V}}{\partial z}\left(\mathbf{V}(x,y)-\mathbf{V}_0\right). \tag{108}$$

Carson (16+17)

Carson explains the meaning of $\mathbf{V}_0$ with the following, and in the next sentence proas: "$\mathbf{V}$(x,y)-$\mathbf{V}_0$ is the scalar potential between the point x,y and the ground, which is due to the charges on the wire and the surface of the ground." In the next line he says. "By means of (16) and the preceding formulas we get:

$$\begin{aligned}\mathbf{E}_{z,A}&=-\left(\frac{\omega\mu}{\pi}\right)I_L\int_0^\infty\left(\left(\eta^2+j\right)^{0.5}-\eta\right)e^{-h'\eta}e^{j\eta x'}e^{y'\left(\eta^2+j\omega\mu\sigma\right)^{0.5}}\,d\eta\\&\quad-\frac{j\omega\mu}{2\pi}I_L\,ln(p''/p')-\frac{\partial \mathbf{V}}{\partial z}\end{aligned} \tag{109}$$

Carson (18)

where

$$\begin{aligned}p'&=\left((h-y)^2+x^2\right)^{0.5} &&\text{Distance of point x,y from wire}\\p''&=\left((h+y)^2+x^2\right)^{0.5} &&\text{Distance of point x,y from underground image of wire.}\end{aligned} \tag{110}$$

"The first two terms on the right hand side of (109) represent the electric force due to the varying magnetic field, the term -∂**V**/∂t represents the axial electric intensity due to charge per unit length." He comments, in one additional sentence, on **V** being calculable from Q, but this is the extent of his review of the equation.

The value of p″ is the distance to an image conductor h distance underground, vertically below the conductor of interest, and p' is the distance to the real conductor, above ground.,

There are confusing aspects to (109):

- The first part of (109) is actually (107), which is the equation that was developed for $\mathbf{E}_{z,G}$, for the electric field in the ground, entering from the air. The physics of using this equation to represent $\mathbf{E}_{z,A}$ are not solidly grasped by this author (JJH).
- The 2nd part (with the log term) appears to be the field one would obtain if the current returned in the image conductor, deep in the ground. This is similar to the image conductor concept used in line to ground capacitance calculations. It models the zero

voltage point as being halfway between the two conductors, i.e., the ground plane, for capacitance calculations, but does not model where the current actually flows. The concept implies a non-conducting earth, but we are obviously not talking about a non-conductive earth.

- Carson is apparently using a superposition concept, but it is not clear how the elements of the equation apply.

Carson now uses this equation to find voltage along the surface of the conductor. Carson states, "Let z denote the "internal" or "intrinsic" impedance of the wire per unit length. (With small error, this may usually be taken as the resistance per unit length of the wire.) The axial electric intensity at the surface of the wire is then zI_L. Equating this to the axial electric intensity at the surface of the wire as given by (109) and replacing $\partial\mathbf{V}/\partial t$ by $-\Gamma$, we have"

$$zI_L = \frac{\omega\mu}{\pi} I_L \int_0^{\infty} \left(\left(\eta^2 + j \right)^{0.5} - \eta \right) e^{-2h'\mu} du - \frac{j\omega\mu}{2\pi} I_L \, log\left(p''/r \right) + \Gamma \mathbf{V} . \quad (111)$$

Carson (19)

Carson makes note that

$$j\omega Q = \Gamma I_L - GV = \Gamma I_L - \frac{G}{C} Q \quad (112)$$

He then reworks (111) and (112) for an equation for Γ giving

$$\Gamma^2 = \left(G + j\omega C \right)\left(R + jX \right)$$
$$\Gamma^2 = \left(G + j\omega C \right)\left[z + \frac{j\omega\mu}{2\pi} ln\left(p''/r \right) + \frac{\omega\mu}{\pi} \int_0^{\infty} \left(\left(\eta^2 + j \right)^{0.5} - \eta \right) e^{-2h'\eta} d\eta \right] . \quad (113)$$

The characteristic impedance of the line, similar to the impedance of the line reviewed in Appendix Section C4, equation (A.88) is

$$Z_C = \left(\frac{\left(R + jX \right)}{\left(G + j\omega C \right)} \right)^{0.5} . \quad (114)$$

The numerator in this equation is the series impedance of the line. Solving (113) for R+jX gives

$$Z_{PP}^{C} = R + jX = z + \frac{j\omega\mu}{2\pi} ln\left(p''/p_r \right) + \frac{\omega\mu}{\pi} \int_0^{\infty} \left(\left(\eta^2 + j \right)^{0.5} - \eta \right) e^{-2h'\eta} d\eta . \quad \text{ohms/meter} \quad (115)$$

Carson (23)

where Z_{PP}^{C} is a reference back to the notation used in section 4a of this paper, p_r is the radius of the conductor, and $z \approx$ wire resistance per Carson (see the paragraph above (111)).

As this is a critical equation in Carson's analysis, let us note again that similar to any equation herein relating current or **H** to voltage, compared to Carson's version, this equation herein has been multiplied by $\mu/4\pi$.

Carson comments on this equation: "It will be observed that the first two terms on the right hand side represent the series impedance of the circuit if the ground is a perfect conductor. The infinite integral formulates the effect of the finite conductivity of the ground."

One of the objectives of this paper is to verify the equation above. However, due to the vague process by which we arrived here, beginning from equation (108) forward, the objective has not been met; hence, the matter will be left to a later update of this paper.

Carson provides a similar equation for the mutual inductance between two overhead wires, but we will not address that equation in this paper.

Analysis of Carson's Series Impedance Equation

Assuming one accepts the above equation (115), it is fairly straightforward to apply 2 out of 3 of the terms, we are left with a quandary about how to analyze the equation's infinite integral. It would be possible to analyze (115) numerically for a variety of h', then find a curve fitting equation that mimics the results. Be aware of the effect of substitution of variables. Recall (106) where $h' = (\omega\mu\sigma)^{0.5} h$ will affect the analysis.

Carson found a somewhat convoluted calculation process to evaluate the integral. The equations are still widely used today, but generally the low-to-moderate accuracy versions are used. The equation process will not be repeated here, since it is rather long. Most textbooks have reproduced simplified versions of the results, and the interested reader can obtain a copy of these equations directly from Carson's paper if one wishes to calculate the matter to high degrees of accuracy.

However, the equation above can be solved numerically, reducing the need to work with Carson's solution process. The calculation is not very intense; Reference [7] is a MS Excel spreadsheet that numerically calculates (115).

Conclusions

The process above may appear fairly involved, but it took many more pages than necessary. Those that wish to plug and chug could reduce all this to a few pages. See [7].

The author would appreciate comments from readers (JJH).

References

1. Carson, John R., Wave Propagation in Overhead Wires With Ground Return, *Bell System Technical Journal*, Vol. 5, 1926. A copy of this paper will be made available at http://home.att.net/~john.horak/.
2. O'Conner, Michael James, The Application of Carson's Equations To Underground Cables, Master of Science Thesis, New Mexico State University, Las Cruces, New Mexico, December 1987. (Contact John Horak for information on copies.)
3. Kersting, William H., Distribution Modeling and Analysis (text), CRC Press, 2002.
4. Anderson, Paul M., Analysis of Faulted Power Systems (text), IEEE/Wiley Press, 1995
5. Edminister, Joseph A, Electromagnetics, 2nd Edition (text), *Schaum's Outline Series*, McGraw-Hill, 1993.
6. Basler Electric (John Horak), "A Derivation of Symmetrical Component Theory and Symmetrical Component Networks," www.basler.com.
7. Basler Electric, ElectricCalcs.xls, MS Excel spreadsheet with some calculations on electrical network related topics. www.basler.com

Biography

John Horak received his BSEE degree from the University of Houston in 1988 and his MSEE degree, specializing in power system analysis, from the University of Colorado, Denver, in 1995. He worked ten years with Stone and Webster Engineering and was on assignment for six years in the System Protection Engineering offices of Public Service Company of Colorado. His work has included fault calculations, relay coordination settings, control design, and equipment troubleshooting. Previous employers include Houston Light and Power and Chevron. John joined Basler Electric in 1997 and is a Senior Application Engineer. He has authored, coauthored, and presented technical papers at a variety of industry events and conferences. John is a member of IEEE-IAS and -PES and has P.E. licenses in Colorado and California. (Contact: johnhorak@basler.com or john.horak@att.net)

Appendix: Some "Basic" Electromagnetic Field Concepts

A) Electromagnetic Field Definitions and Relationships

The material is intended to help with the interpretation of the analysis of the paper. It consolidates some of the concepts that one might find scattered in various textbooks. However, it cannot substitute for a dedicated in-depth text. A good but inexpensive resource on the topic is Schaum's Outline on Electromagnetics [4].

The paper uses the SI-MKS unit system, but Carson's paper was developed in a CGS unit system. One purpose of this appendix is to allow one to compare the equations in the two systems. The table below defines the symbols associated with electromagnetic fields, the units in the SI-MKS system, and some auxiliary relationships, as used in the paper.

Table A.1 - Symbols, Terms, Units, SI-MKS System

Symbol	Term	Units	Notes; additional relationships
E	Electric Field Strength	Volts / meter (V/m)	1 Volt(V) = 1 Watt(W) / Amp(A); 1W = 1 Joule(J) / second(s); 1J = 1 Newton(N)·m; $V = \int E\,dx$
D	Electric Flux Density	Coulombs / meter^2 (C/m^2)	1 Coulomb of flux = electric flux that radiates from (or to) 1 coulomb of charge.
H	Magnetic Field Strength	Amps / meter (A/m)	1A = current that will cause a force of $2\cdot 10^7$N on 2 wires 1 meter apart
B	Magnetic Flux Density	Teslas (T) or Weber (Wb) / m^2	1 Tesla = 1 Wb / m^2 1 Weber (Wb) = 1 V·s. If d(Wb)/dt = 1 Wb / s, then V = 1Volt.
$\mathbf{J_C}$	Conduction Current	Amps / meter^2 (A/m^2)	Refers to charged particle movement, e.g., electrons, ions, "holes"
$\mathbf{J_C}$	Displacement Current	Amps / meter^2 (A/m^2)	Refers to effective current due to changing electric flux density, e.g. space between capacitor plates
m	length	meter (m)	1 meter = distance light travels in (1/299,792,458)s in a vacuum.
Q	Electric Charge	Coulomb (C) or A·s	1 Coulomb of electrons = 1A for 1s $\approx 6.24\times 10^{18}$ electrons. (Note: 1C ≠ 1 mole)

ρ	Charge Density	Coulombs / meter^3 (C/m^3)	ρ is also frequently used as the symbol for resistivity; ρ always refers to charge density in the paper.
σ	Conductivity	Siemens / meter (S/m) or A / (V·m)	σ = conductivity in Siemens. σ = G·Length /Area
G R	Conductance Resistance	Siemens (S) Ohms (Ω)	G = conductance in Siemens 1 S = 1 A / V = 1/R

The table below lists the equations and relationships between electric and magnetic fields that are commonly referred to as Maxwell's equations. (Actually, Maxwell knew of these relationships, but might not immediately recognize these equations. He wrote his equations in a somewhat different format.) (The terms $\nabla \times \mathbf{E}$, $\nabla \times \mathbf{H}$ (curl equations), $\nabla \bullet \mathbf{D}$, and $\nabla \bullet \mathbf{B}$ (divergence equations) are covered in Appendix Part B.)

Table A.2 - Maxwell's Electromagnetic Field Equations, SI-MKS System

Common Name	Differential Form	Integral Form
Faraday's Law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (A.1)	$\mathbf{V} = \int_{CL} \mathbf{E} \bullet d\mathbf{l} = \int_{S} \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \bullet d\mathbf{s}$ (A.2)
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J_C} + \mathbf{J_D}$ (A.3) $\mathbf{J_C} = \sigma \mathbf{E}$ (A.4) $\mathbf{J_D} = \frac{\partial \mathbf{D}}{\partial t}$ (A.5)	$\int_{CL} \mathbf{H} \bullet d\mathbf{l} = \int_{S} (\mathbf{J_C} + \mathbf{J_D}) \bullet d\mathbf{s}$ (A.6)
Gauss' Law	$\nabla \bullet \mathbf{D} = \rho$ (A.7)	$\int_{SV} \mathbf{D} \bullet d\mathbf{s} = \int_{V} \rho \, dv$ (A.8)
"No magnetic monopole"	$\nabla \bullet \mathbf{B} = 0$ (A.9)	$\int_{SV} \mathbf{B} \bullet d\mathbf{s} = 0$ (A.10)

In the integral form of the equations:

- CL indicates an integral along closed line CL; this line defines the boundary of surface S
- S indicates an integral over surface S, where the surface does not enclose a volume, and whose boundary is the closed line CL
- SV indicates an integral over a surface that completely encloses a volume V
- V indicates an integral of all elements in the volume V enclosed by surface SV

Note how **V** and **E** are used in tables A.1 and A.2. Be aware that in power system studies the letter E sometimes is used to indicate volts, but in electromagnetic studies E usually indicates volts/meter.

Note in Table 2 there are no constants in the equations. This set of equations for stating Maxwell's equations is chosen for that very reason. There are, however, equation constants needed when the relationship between D and E, and B and H, and E and J_C are introduced. These relationships are as follows:

Table A.3 - Other Relationships: B/H, E/D, Force, E_{Static}, SI-MKS System

Relationship	Notes
$\mathbf{B} = \mu\mathbf{H}$ $\mu = \mu_0\mu_r$ (A.11)	$\mu_0 = 4\pi 10^{-7}\ \frac{Wb}{A \cdot m}$ (Permeability) This is the exact defined value for μ_0 per SI System $\mu_r = 1$ in a vacuum, rises past 10^3 for ferromagnetic materials
$\mathbf{E} = \varepsilon\mathbf{D}$ $\varepsilon = \varepsilon_0\varepsilon_r$ (A.12)	$\varepsilon_0 = 8.854188 \cdot 10^{-12}\ \frac{C}{V \cdot m}$ (Permittivity) This is an approximation of ε_0, which is calculated from other defined constants. $\varepsilon_r = 1$ in a vacuum, rises as high as 5 in some linear dielectrics, and rises past 10^5 for certain non-linear dielectrics
$\mathbf{F} = Q(\mathbf{E} + \mathbf{U} \times \mathbf{B})$ (A.13)	F = force in Newtons; Q = charge in coulombs; U = velocity in m/s
$\mathbf{E}_{Static} = \frac{Q}{4\pi\varepsilon r^2}$ (A.14)	Field due to isolated single point charge. Direction is radial.

Comparing SI-MKS Units to CGS Unit Systems

Carson's work was prepared in an era before the SI-MKS system was developed, and to understand his work, one has to have at least some concept of the CGS unit system in use at the time. In the CGS system, distance is in centimeters, mass is in grams, force is in dynes (1 dyne = 10^{-5} Newtons), work is in ergs (1 erg = 10^{-7} joules), and the speed of light is generally stated in cm/s. However, there are several variations of CGS units for other quantities, especially in regard to electromagnetic fields. Various unit systems that have been in use include the CGS esu system (used by the early pioneers of electric field studies), the CGS emu system (used by the early pioneers in magnetic field studies), the CGS Gaussian system (a composite of the esu and emu system), and the Heaviside-Lorentz system (a variation of the Gaussian system that changed how a 4π factor appears in the equations). Each of these systems has its own units for **B**, **H**, **E**, **D**, **F**, ε, μ, ρ, σ, etc. and may have variations in how a 4π factor shows up in the equations. The example below shows the equations for one variation of the CGS system, but one cannot say for certain which CGS system it refers to without delving into the units involved. Further, in some presentations of CGS systems, the c factors shown in the table might not exist and may instead be incorporated into the units used in the system.

Table A.4 - Example CGS System Equations (units change depending on variation of CGS system in use, and c constant may be incorporated into units)

Maxwell's Equations, CGS, typical		
Differential Form	Integral Form	Comments
$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ (A.15)	$\int_{CL}\mathbf{E}\bullet d\mathbf{l} = \frac{1}{c}\int_{S}\left(-\frac{\partial \mathbf{B}}{\partial t}\right)\bullet d\mathbf{s}$ (A.16)	c is $2.9979\cdot 10^{10}$ cm/s
$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J_C} + \frac{1}{c}\mathbf{J_D}$ (A.17) $\mathbf{J_D} = \frac{\partial \mathbf{D}}{\partial t}$	$\int_{CL}\mathbf{H}\bullet d\mathbf{l} = \frac{1}{c}\int_{S}(4\pi\mathbf{J_C} + \mathbf{J_D})\bullet d\mathbf{s}$ (A.18)	The 4/π factor in $\int \mathbf{J_C}$ says that H per meter around conductor = 2I/r, not I/2πr as in the MKS system
$\nabla\bullet\mathbf{D} = 4\pi\rho$ (A.19)	$\int_{SV}\mathbf{D}\bullet d\mathbf{s} = 4\pi\int_{V}\rho\, dv$ (A.20)	The 4π factor also appears in a polarizing index associated with k_M and k_E, below
$\nabla\bullet\mathbf{B} = 0$ (A.21)	$\int_{SV}\mathbf{B}\bullet d\mathbf{s} = 0$ (A.22)	
B, **E**, **F** Relationships, CGS, typical		
$\mathbf{B} = k_M\mathbf{H}$ (A.23) $k_M = (1 + 4\pi X_M)$	X_M =0 in vacuum (making k_M = 1) and >1 in magnetizable media. Note k_M needs units to match **B** to **H**, even if = 1.	
$\mathbf{E} = k_E\mathbf{D}$ (A.24) $k_E = (1 + 4\pi X_E)$	X_E =0 in vacuum (making k_E = 1), and >1 in polarizable media. Note k_E needs units to match **E** to **D**, even if = 1.	
$\mathbf{F} = Q\left(\mathbf{E} + \frac{\mathbf{U}\times\mathbf{B}}{c}\right)$ (A.25)	In some variations of this equation, depending on units used for Q and **E**, the vector **B** might be replaced by **H**.	

B) Equations for $\nabla\times(\ )$, $\nabla\bullet(\ )$, and $\nabla^2(\ )$

In tracing out the analysis in the paper, it may be helpful to see these "del" (also called "nabla") equations written out. The equations below are for a right handed Cartesian coordinate system. The basic equations to which we will apply the ∇ operator are (39) and (40) of the main text, but also include the similar **D** and **B** field equations.

The $\nabla\times$ operator applied to the **E** and **H** fields is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{fully written out is:} \quad \begin{aligned} \frac{\partial \mathbf{E}_z}{\partial y} - \frac{\partial \mathbf{E}_y}{\partial z} &= -\frac{\partial \mathbf{B}_x}{\partial t}\hat{x} \\ \frac{\partial \mathbf{E}_x}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial x} &= -\frac{\partial \mathbf{B}_y}{\partial t}\hat{y} \ . \\ \frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} &= -\frac{\partial \mathbf{B}_z}{\partial t}\hat{z} \end{aligned} \tag{A.26}$$

$$\nabla \times \mathbf{H} = \mathbf{J_C} + \mathbf{J_D} \quad \text{fully written out is:} \quad \begin{aligned} \frac{\partial \mathbf{H}_z}{\partial y} - \frac{\partial \mathbf{H}_y}{\partial z} &= \mathbf{J}_{\mathbf{C},x}\hat{x} + \frac{\partial \mathbf{D}_x}{\partial t}\hat{x} \\ \frac{\partial \mathbf{H}_x}{\partial z} - \frac{\partial \mathbf{H}_z}{\partial x} &= \mathbf{J}_{\mathbf{C},y}\hat{y} + \frac{\partial \mathbf{D}_y}{\partial t}\hat{y} \ . \\ \frac{\partial \mathbf{H}_y}{\partial x} - \frac{\partial \mathbf{H}_x}{\partial y} &= \mathbf{J}_{\mathbf{C},z}\hat{z} + \frac{\partial \mathbf{D}_z}{\partial t}\hat{z} \end{aligned} \tag{A.27}$$

(or, re-write with $\mathbf{J}_C = \sigma\mathbf{E}$)

The $\nabla \bullet$ operator applied to the **D** and **B** fields is

$$\nabla \bullet \mathbf{D} = \rho \quad \text{fully written out is:} \quad \frac{\partial \mathbf{D}_x}{\partial x} + \frac{\partial \mathbf{D}_y}{\partial y} + \frac{\partial \mathbf{D}_z}{\partial z} = \rho \, . \tag{A.28}$$

$$\nabla \bullet \mathbf{B} = 0 \quad \text{fully written out is:} \quad \frac{\partial \mathbf{B}_x}{\partial x} + \frac{\partial \mathbf{B}_y}{\partial y} + \frac{\partial \mathbf{B}_z}{\partial z} = 0 \, . \tag{A.29}$$

The ∇^2 operator (sometimes called the Laplacian) has similar but different meanings for a vector and a scalar quantity. Let us apply the ∇^2 to a vector by writing out the (A.51) **E** field wave equation, described in section C below.

$$\nabla^2 \mathbf{E} = \gamma^2 \mathbf{E} \quad \text{fully written out is:} \quad \begin{aligned} \left(\frac{\partial^2 \mathbf{E}_x}{\partial x^2} + \frac{\partial^2 \mathbf{E}_x}{\partial y^2} + \frac{\partial^2 \mathbf{E}_x}{\partial z^2}\right) &= \gamma^2 \mathbf{E}_x \hat{x} \\ \left(\frac{\partial^2 \mathbf{E}_y}{\partial x^2} + \frac{\partial^2 \mathbf{E}_y}{\partial y^2} + \frac{\partial^2 \mathbf{E}_y}{\partial z^2}\right) &= \gamma^2 \mathbf{E}_y \hat{y} \ . \\ \left(\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + \frac{\partial^2 \mathbf{E}_z}{\partial z^2}\right) &= \gamma^2 \mathbf{E}_z \hat{z} \end{aligned} \tag{A.30}$$

where γ is the propagation constant, also described below. For the **H** equations

$$\nabla^2 \mathbf{H} = \gamma^2 \mathbf{H} \quad \text{fully written out is:} \quad \begin{aligned} &\left(\frac{\partial^2 \mathbf{H}_x}{\partial x^2} + \frac{\partial^2 \mathbf{H}_x}{\partial y^2} + \frac{\partial^2 \mathbf{H}_x}{\partial z^2}\right) = \gamma^2 \mathbf{H}_x \hat{x} \\ &\left(\frac{\partial^2 \mathbf{H}_y}{\partial x^2} + \frac{\partial^2 \mathbf{H}_y}{\partial y^2} + \frac{\partial^2 \mathbf{H}_y}{\partial z^2}\right) = \gamma^2 \mathbf{H}_y \hat{y} \\ &\left(\frac{\partial^2 \mathbf{H}_z}{\partial x^2} + \frac{\partial^2 \mathbf{H}_z}{\partial y^2} + \frac{\partial^2 \mathbf{H}_z}{\partial z^2}\right) = \gamma^2 \mathbf{H}_z \hat{z} \end{aligned} \tag{A.31}$$

Applying ∇^2 to a scalar, assume that ρ (charge density) varies with x, y, and z. For this condition

$$\nabla^2 \rho = \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}. \tag{A.32}$$

C) Electromagnetic Wave Equations, Time Harmonic Format

C.1 - Exponential Notation

Complex number exponential terminology is used to simplify analysis when there are time derivatives and integrals involving sines and cosines of ωt, where ω=2πf. The paper uses exponential notation extensively. As a reminder, one is working with the real part of the exponential equation; i.e.: *"In the final real world application of the equation, take the real part of the equation and drop the imaginary part."* The "*Re*" symbol, used below, implies this concept, and is generally left as implied.

First recall that

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t). \tag{A.33}$$

Hence, when expressing a sine and cosine as a form of $e^{j\omega t}$, one needs to be aware that

$$\cos(\omega t) = Re\left(e^{j\omega t}\right) \tag{A.34}$$

$$\sin(\omega t) = Re\left(e^{j(\omega t - \pi/2)}\right) = Re\left(-je^{j\omega t}\right). \tag{A.35}$$

Note that a sine function is indicated by adding a -jπ/2 factor to the exponent or multiplying by -j. Also, complex numbers multiplied by an $e^{j\omega t}$ term are adding a phase shift. The process can be seen by this example:

$$\begin{aligned} &A = 1.732 + j1.0 = 2\angle 30^0 = 2\angle 0.5236\text{rad} \\ &B = e^{j\omega t} \\ &C = A \cdot B = Re\left((1.732 + j1.0)e^{j\omega t}\right) = Re\left(2e^{j\omega t + 0.5236}\right) = 2\cos(\omega t + 0.5236) \end{aligned} \tag{A.36}$$

The derivative and integral of $e^{j\omega t}$ follows the same rule as if real numerators were in the exponent. As an example, assume the term we start with is a cosine modulated at 60hz with a +30 deg shift, which in radians is A = cos(ωt+0.5236) where ω=2π60.

$$A = 2\,cos(\omega t + 0.5236) = Re\left(e^{j(\omega t+0.5236)}\right)$$

$$\frac{dA}{dt} = \frac{d}{dx}e^{j(\omega t+0.5236)} = Re\left(j\omega 2e^{j(\omega t+0.5236)}\right) = \omega 2\,cos\left(\omega t + \frac{\pi}{2} + 0.5236\right) \quad (A.37)$$

$$\int A\,dt = \int e^{j(\omega t+0.5236)}dt = Re\left(\frac{2}{j\omega}e^{j(\omega t+0.5236)}\right) = \frac{2}{\omega}cos\left(\omega t - \frac{\pi}{2} + 0.5236\right)$$

The value of this process is in the conciseness with which cosine and sine terms and their 1st and 2nd derivatives with respect to time can be stated and merged with equations that already contain complex number, phase shift, and e^{bx} type terms. For instance, compare these two columns:

$$A = (1.732 + j1)e^{bx}\,cos(\omega t + \varphi)$$
$$= 2e^{bx}\,cos(\omega t + \varphi + 0.5236)$$
$$1\angle -90A = 2e^{bx}\,cos(\omega t + \varphi + 0.5236 - \pi/2)$$
$$= 2e^{bx}\,sin(\omega t + \varphi + 0.5236)$$
$$\frac{dA}{dt} = -2\omega\left(e^{bx}\,sin(\omega t + \varphi + 0.5236)\right)$$
$$\frac{d^2A}{dt^2} = -2\omega^2\left(e^{bx}\,cos(\omega t + \varphi + 0.5236)\right)$$
$$\int Adt = \frac{2}{\omega}\left(e^{bx}\,cos(\omega t + \varphi + 0.5236)\right)$$

$$A = (1.732 + j1)e^{bx+j\omega t}$$
$$= 2e^{bx+j(\omega t+0.5236)}$$
$$1\angle -90A = 2e^{bx+j(\omega t+0.5236-\pi/2)}$$
$$= -jA = (1/j)A$$
$$\frac{dA}{dt} = j\omega A$$
$$\frac{d^2A}{dt^2} = (j\omega)^2 A = -\omega^2 A$$
$$\int Adt = \frac{1}{j\omega}A = \frac{-j}{\omega}A$$

Both sides imply the same net equations. The right side approach is much easier to work with when equations become very extensive.

C.2 - Electromagnetic Field Wave Equations

A substantial level of credit for the analysis process below needs to be given to chapter 14 of [4].

In linear uniform material where charge density is uniform or 0,

$$\nabla \bullet \mathbf{D} = 0 \quad \text{and} \quad \mathbf{J_C} = \sigma\mathbf{E}\,. \quad (A.38)$$

For these conditions, Maxwell's equations (Table A.2) can be stated in a time harmonic form that is only in terms of **E** and **H**:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (A.39)$$
$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E} \quad (A.40)$$
$$\nabla \bullet \mathbf{E} = 0 \quad (A.41)$$
$$\nabla \bullet \mathbf{H} = 0 \quad (A.42)$$

Taking the curl of (A.39) and (A.40) gives

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\nabla \times \mathbf{H}) \tag{A.43}$$

$$\nabla \times (\nabla \times \mathbf{H}) = (\sigma + j\omega\mu)(\nabla \times \mathbf{E})\,. \tag{A.44}$$

In Cartesian coordinates, the term on the right side can be restated as

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} \tag{A.45}$$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla(\nabla \bullet \mathbf{H}) - \nabla^2 \mathbf{H}\,. \tag{A.46}$$

Since from (A.41) and (A.42) we said $\nabla \bullet \mathbf{E} = 0$ and $\nabla \bullet \mathbf{H} = 0$, these last two equations reduce to

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} \tag{A.47}$$

$$\nabla \times (\nabla \times \mathbf{H}) = -\nabla^2 \mathbf{H}\,. \tag{A.48}$$

Now, substitute (A.40) and (A.47) into (A.43), and then substitute (A.39) and (A.48) into (A.44). The resultant equations are

$$\nabla^2 \mathbf{E} = j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E} \tag{A.49}$$

$$\nabla^2 \mathbf{H} = j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{H}\,. \tag{A.50}$$

or

$$\nabla^2 \mathbf{E} = \gamma^2 \mathbf{E} \tag{A.51}$$

$$\nabla^2 \mathbf{H} = \gamma^2 \mathbf{H} \tag{A.52}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)\,. \tag{A.53}$$

The value for γ is called the propagation constant. The term refers to the nature of the traveling wave described by this equation. We will cover this more deeply below.

C.3 - Plane Wave Propagation

Now let us see how to apply this equation by analyzing the equations of a plane wave. Recall the basic form of the equation for **E** as stated in (41). Now assume a uniform **E** field that is oriented in the +/- $\hat{x}$ direction, where the magnitude is oscillating in a sinusoidal pattern over time, and where an observer on an x-y plane, stationary with respect to $\hat{z}$, sees the same **E** field for any value of x and y. One might look forward to figure A.1 to see the wave equation that we are going to describe. For this field, (41) reduces to

$$\mathbf{E} = \mathbf{E}_x(z)e^{j(\omega t+\varphi)}\hat{x}\,. \tag{A.54}$$

where the equation for $\mathbf{E_x}(z)$ has not been determined. Note that $\mathbf{E}_x\hat{x}$ is not a function of x or y, so the **E** field is the same for all x and y. To find the equation, let us apply (A.54) to (A.51) as interpreted by (A.30). The equation that will result is

$$\gamma^2 \mathbf{E}_x = \left(\frac{\partial^2 \mathbf{E}_x(z)}{\partial z^2}\right)e^{j(\omega t+\varphi)}\hat{x}\,. \tag{A.55}$$

The equation for $\mathbf{E_x}$ that agrees with (A.55) is:

$$\mathbf{E}_x = \left(Ae^{j(\omega t+\varphi)\pm\gamma z}\right)\hat{x}\,. \qquad \text{(A.56)}$$

The ± in front of γz is applied as follows: For positive z, we use negative γz, and for negative z, we use the positive γz. The resulting equation describes a traveling wave, to be seen below.

From (A.53) we have additional information on valid data for γ. We can see that for a conductive media, such as the ground or copper, γ takes on a complex number format. The real and imaginary parts of γ are

$$\gamma = \alpha + j\beta \quad \text{(units are 1/meters, or m}^{-1}\text{)} \qquad \text{(A.57)}$$

$$\alpha = \omega\left[\frac{\mu\varepsilon}{2}\left(\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{0.5}-1\right)\right]^{0.5} \qquad \text{(A.58)}$$

$$\beta = \omega\left[\frac{\mu\varepsilon}{2}\left(\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{0.5}+1\right)\right]^{0.5}. \qquad \text{(A.59)}$$

There are several different forms of (A.56) that are insightful:

$$\mathbf{E}_x = \left(Ae^{j(\omega t+\varphi)-\gamma z}\right)\hat{x} \qquad \text{(Using -}\gamma z\text{ for +z)} \qquad \text{(A.60)}$$

$$\mathbf{E}_x = \left(Ae^{j(\omega t+\varphi)-(\alpha+j\beta)z}\right)\hat{x} \qquad \text{(Showing complex }\gamma\text{)} \qquad \text{(A.61)}$$

$$\mathbf{E}_x = \left(Ae^{j(\omega t+\varphi-(\beta-j\alpha)z)}\right)\hat{x} \qquad \text{(Putting j before the entire exponent)} \qquad \text{(A.62)}$$

$$\mathbf{E}_x = \left(Ae^{j(\omega t+\varphi-\beta z)}e^{-\alpha z}\right)\hat{x} \qquad \text{(Showing the exponential decay component)} \qquad \text{(A.63)}$$

$$\mathbf{E}_{x(\sigma=0,\varphi=0)} = \left(Ae^{j(\omega t-\beta z)}\right)\hat{x} \qquad \text{(The case where }\sigma = 0\text{ and }\varphi = 0\text{)} \qquad \text{(A.64)}$$

Some papers tend to use a slightly different definition of the propagation constant. Let us name it P where

$$P = \beta - j\alpha = \frac{\gamma}{j} \qquad \text{(A.65)}$$

Now we can restate (A.62) as

$$\mathbf{E}_x = \left(Ae^{j(\omega t+\varphi-Pz)}\right)\hat{x} \quad \text{or, if } \varphi = 0\text{:} \quad \mathbf{E}_x = \left(Ae^{j(\omega t-Pz)}\right)\hat{x} \qquad \text{(A.66)}$$

Note we have moved the j to encompass the entire exponent and, in the process, the real and imaginary parts of γ have been changed as seen in (A.65). When equations of the format of (A.64) or (A.66) are seen in papers or texts, if the material is referring to a non-conductive media, then the format of (A.64) is inferred. If the context refers to a conductive media, then the format of (A.66) is inferred.

In free space, $\sigma=0$, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, and α, and β reduce to

$$\alpha = 0 \tag{A.67}$$

$$\beta = \omega\left(\mu_0\varepsilon_0\right)^{0.5}. \tag{A.68}$$

Hence

$$\gamma = j\omega\left(\mu_0\varepsilon_0\right)^{0.5}. \tag{A.69}$$

This could also be seen just by inspection of (A.53) and setting $\sigma = 0$.

Note we are close to describing the equation for speed of light in free space. Let

$$c = \frac{\omega}{\beta} \quad \text{which in free space becomes:} \quad c = \frac{1}{\left(\mu_0\varepsilon_0\right)^{0.5}}. \tag{A.70}$$

The propagation speed in meters / second is defined by c. One should see that in free space (A.63) may be restated as

$$\mathbf{E}_z = \left(Ae^{j\omega\left(t+\frac{\varphi}{\omega}-\frac{\beta z}{\omega}\right)} \right)\hat{x} = \left(Ae^{j\omega\left(t+\frac{\varphi}{\omega}-\frac{z}{c}\right)} \right)\hat{x}. \tag{A.71}$$

The traveling wave nature of the wave is seen in (A.71). If one moves in the $\hat{z}$ direction fast enough so that $z/c = t$, then the exponent in (A.71) does not change with time; hence, one is moving at the speed of light in free space. The speed of light in free space is about 3e8 m/s or 186,000 miles/s.

An associated term is wavelength, which is the answer to "In one cycle ($\Delta t = 2\pi/\omega$), how far has the observer on the $\hat{z}$ axis moved?" Wavelength typically is given the symbol λ and has the equations

$$\lambda = \frac{c}{\omega/2\pi} = \frac{c}{f} = \frac{2\pi}{\beta}. \tag{A.72}$$

The wavelength of a 60Hz signal in free space is about 5000km, or about 3100 miles. On a metallic wire, however, the wavelength is a couple percent shorter. See section C.4.

We need to be aware of how the above equations appear in a conductive media. Let us define a conductive material to be a case where

$$\frac{\sigma}{\omega\varepsilon} \gg 1. \tag{A.73}$$

At 60Hz it does not take much conductivity for (A.73) to be true. For comparison, $\varepsilon_0 \approx 8.85e\text{-}12$ C/Vm, so at 60hz, $\omega\varepsilon_0 \approx 3.34e\text{-}9$ S/m. Comparing to σ of copper, ≈ 5.8e7 S/m, typical assumed earth resistivity, ≈ 100 S/m and typical drinking water, ≈ 0.02 S/m, we see all of these materials make (A.73) true. When (A.73) is true

$$\alpha \approx \beta \approx (0.5\omega\mu\sigma)^{0.5}. \tag{A.74}$$

Hence

$$\gamma \approx (0.5\omega\mu\sigma)^{0.5}(1+j1). \tag{A.75}$$

These quantities for α, β, and γ will reduce propagation speed, shorten wavelength, and cause an exponential decay of the wave magnitude, as seen in (A.63). Some informative calculations:

- In copper at 60Hz and σ = 5.8e7 S/m:
  $\alpha \approx \beta \approx 117\ m^{-1}$, c ≈ 3.22 m/s, wavelength ≈ 0.0536 m, and $e^{-\alpha z} = e^{-1}$ at ≈ 0.0085 m
- In earth, at 60Hz and σ = 100 S/m:
  $\alpha \approx \beta \approx 0.154\ m^{-1}$, c ≈ 2450 m/s, wavelength ≈ 40.8 m, and $e^{-\alpha z} = e^{-1}$ at ≈ 6.50 m
- In drinking water at 60Hz, and σ = 5.8e7
  $\alpha \approx \beta \approx 0.00218\ m^{-1}$, c ≈ 1.73e5 m/s, wavelength ≈ 2890 m, and $e^{-\alpha z} = e^{-1}$ at ≈ 460 m.

An important concept is that one has limited ability to maintain an electric field within a conductor, which is the nature of how power is transmitted through copper and other conductors. When flux penetrates a wire, it generates a voltage along the wire. The wire acts as a wave guide, transmitting the electric field to the remote ends of the wire.

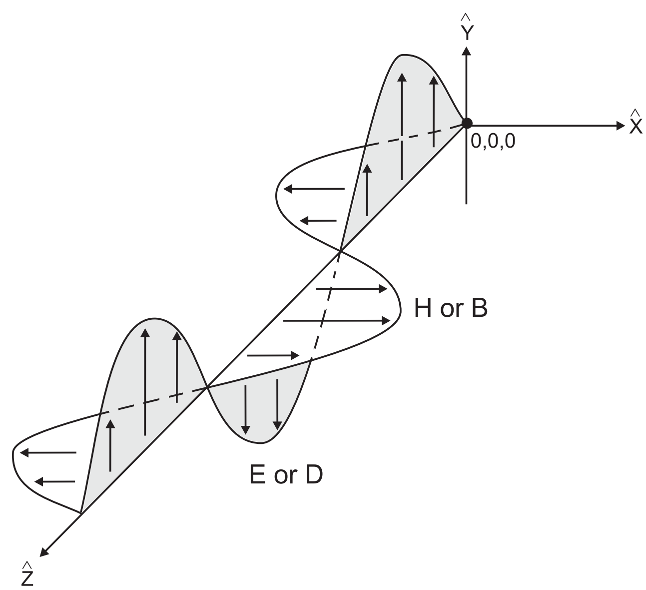

Figure A.1 - A Traveling Electromagnetic Field

C.4 - Wave Propagation in a Wire, with Wire Return

Assume two wires configured per figure A2. The wires are strung on the $\hat{z}$ axis, and the wires are vertically oriented at y_1 and y_2 above ground, and assume the ground is relatively remote so that it's effects on the wave propagation is small. Assume V_0 and I_0 at z = 0. Assume there is no reflected wave from the remote end. Let v(z) be the magnitude of the voltage between the two wires at point z. Similarly, i(z) is the current in the wire at point z.

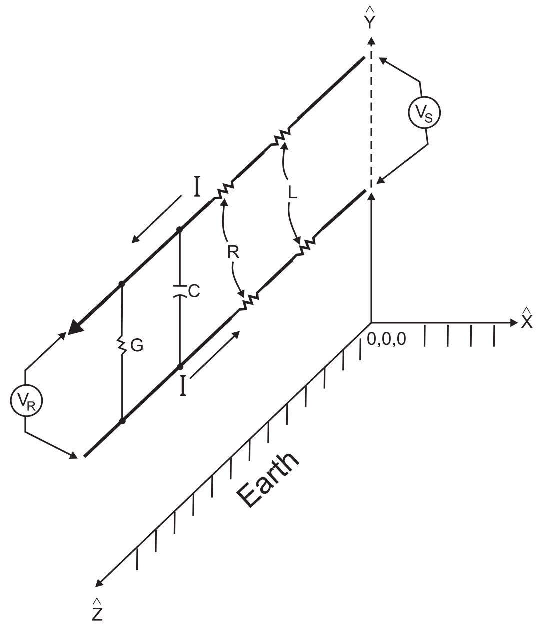

Figure A.2 - Two Overhead Wires, Vertical Construction

Along the line the voltage between the two wires is:

$$v_{z,t} = v_z e^{j\omega t} \tag{A.76}$$

$$i_{z,t} = i_z e^{j\omega t}. \tag{A.77}$$

where v_z and i_z are functions that describe voltage and current in terms of position z. Recall from electric circuit theory that, as we go farther from the source, the voltage and current decreases according to the relationships

$$\frac{\partial v_{z,t}}{\partial z} = -(R + j\omega L)\, i_{z,t} \tag{A.78}$$

$$\frac{\partial i_{z,t}}{\partial z} = -(G + j\omega C)\, v_{z,t}. \tag{A.79}$$

Now take the derivative of (A.78) and (A.79) with respect to z

$$\frac{\partial^2 v_{z,t}}{\partial z^2} = -(R + j\omega L)\frac{\partial i_{z,t}}{\partial z} \\ = (R + j\omega L)(G + j\omega C)\, v_{z,t} \tag{A.80}$$

$$\frac{\partial^2 i_{z,t}}{\partial z^2} = -(G + j\omega C)\frac{\partial v_{z,t}}{\partial z} \\ = (R + j\omega L)(G + j\omega C)\, i_{z,t} \quad . \tag{A.81}$$

These equations can be reduced to a format very much the same as (A.51) and (A.52), but with a different equation for the propagation constant

$$\frac{\partial^2 v_{z,t}}{\partial z^2} = \gamma^2 v_{z,t} \tag{A.82}$$

$$\frac{\partial^2 i_{z,t}}{\partial z^2} = \gamma^2 i_{z,t} \tag{A.83}$$

$$\gamma = \alpha + j\beta = \left((R + j\omega L)(G + j\omega C)\right)^{0.5} . \tag{A.84}$$

The equation format of $Ae^{\gamma z}$ agrees with (A.82) and (A.83), but where γ could be positive or negative. Examining just the voltage equation, the range of possible solutions takes on the format

$$v_{z,t} = \left(A_1 e^{\gamma z} + A_2 e^{-\gamma z}\right) e^{j\omega t} . \tag{A.85}$$

Hence

$$\frac{\partial v_{z,t}}{\partial z} = \gamma\left(A_1 e^{\gamma z} - A_2 e^{-\gamma z}\right) e^{j\omega t} . \tag{A.86}$$

Substituting (A.86) into (A.78) and simplifying

$$i_{z,t} = \frac{1}{Z_c}\left(-A_1 e^{\gamma z} + A_2 e^{-\gamma z}\right) e^{j\omega t} . \tag{A.87}$$

The characteristic line impedance is defined as

$$Z_c = \left(\frac{(R + j\omega L)}{(G + j\omega C)}\right)^{0.5} \tag{A.88}$$

At $z = 0$, $v = V_0$, $i = I_0$, and $t=0$, (A.85) and (A.87), reduce to

$$V_0 = (A_1 + A_2) \\ I_0 = \frac{1}{Z_c}(-A_1 + A_2) .$$

Solving for A_1 and A_2 gives

$$A_1 = \frac{V_0 - I_0 Z_c}{2} \tag{A.89}$$

$$A_2 = \frac{V_0 + I_0 Z_c}{2}. \tag{A.90}$$

Hence, to find the voltage and current some distance from the sending end, for known voltage and current at the sending end, apply (A.89) and (A.90) to (A.85) and (A.87). Some texts report these last equations in terms of knowing V_0 and I_0 at the receiving end, and this changes the definition of the equations a bit.

Speed of Propagation on the Wire Pair

In almost all basic texts on power system analysis there is a derivation of L and C for a pair of wires, so the derivation is not repeated here. The net equation is, for spacing D between two conductors, actual conductor radius r_a of both conductors, and effective conductor radius (for inductance calculations) of r_e

$$L = \frac{\mu}{\pi} ln \frac{D}{r_e} \quad \text{H/m} \tag{A.91}$$

$$C = \frac{\pi\varepsilon}{ln \frac{D}{r_a}} \quad \text{F/m}. \tag{A.92}$$

In (A.84), if R = 0 and G = 0, and then (A.91) and (A.92) are substituted into (A.84), the corresponding γ and the speed of propagation reduces to

$$\gamma = \beta = \left(-\omega^2 LC\right)^{0.5} = j\omega\left(\mu\varepsilon\right)^{0.5} \left(\frac{ln \frac{D}{r_e}}{ln \frac{D}{r_a}} \right)^{0.5}. \tag{A.93}$$

Comparing to the equations of propagation in free space (A.70), and inserting some typical conductor spacing and radii, one will find for the typical spacing of overhead power system lines that

$$c_{wire,lossless} = (0.97 \text{ to } 0.98) \times c_{Free\ Space}. \tag{A.94}$$

This implies a wavelength that is proportionately shorter than the free space wavelength of light. Also, in cable construction, the speed ratio as calculated above can fall to the 0.9-0.95 range, but capacitance and inductance is harder to calculate exactly due to the proximity of ground and the small D/r that is involved.